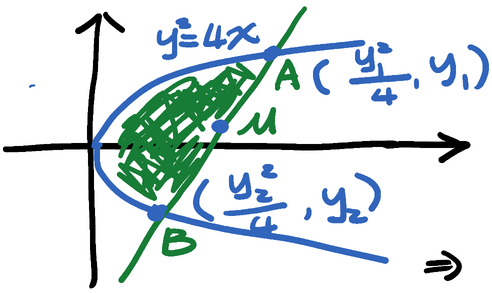
11. 一拋物線 $y^2=4x$ 與一直線交於 A、B 兩點，已知拋物線與直線所圍出來的面積為 $\frac{9}{8}$，則 A、B 的中點軌跡方程式為______。



(Sol) 假設如左：

$$m_{\overline{AB}} = \frac{y_1-y_2}{\frac{y_1^2}{4}-\frac{y_2^2}{4}} = \frac{4}{y_1+y_2}$$

$\overline{AB}$ 的方程式：

$$\frac{4}{y_1+y_2} = \frac{y-y_1}{x-\frac{y_1^2}{4}}$$

$$\frac{4}{y_1+y_2}\left(x-\frac{y_1^2}{4}\right) = y-y_1,$$

$$x-\frac{y_1^2}{4} = \frac{y-y_1}{4}(y_1+y_2)$$

$$x = \frac{y_1+y_2}{4}y + \frac{y_1^2-y_1^2-y_1y_2}{4}$$

$$x = \frac{y_1+y_2}{4}y + \frac{-y_1y_2}{4}$$

$$\Rightarrow x = \frac{y_1+y_2}{4}y - \frac{y_1y_2}{4}$$

虛線面積 $=\frac{9}{8}$

$$\Rightarrow \int_{y_2}^{y_1} \left(\frac{y_1+y_2}{4}y - \frac{y_1y_2}{4}\right) - \frac{y^2}{4}\,dy = \frac{9}{8}$$

$$\Rightarrow -\frac{y^3}{12} + \frac{(y_1+y_2)}{8}y^2 - \frac{y_1y_2}{4}y \Big|_{y_2}^{y_1} = \frac{9}{8}$$

$$\Rightarrow -\frac{y_1^3-y_2^3}{12} + \frac{(y_1+y_2)}{8}(y_1^2-y_2^2) - \frac{y_1y_2(y_1-y_2)}{4} = \frac{9}{8}$$

$\times 24$

$$\Rightarrow (y_1-y_2)\left(-2(y_1^2+y_1y_2+y_2^2)+3y_1^2+\cancel{6y_1y_2}+3y_2^2-\cancel{6y_1y_2}\right) = 27$$

$$\Rightarrow (y_1-y_2)^3 = 27 \Rightarrow y_1-y_2 = 3$$

由 $y_1-y_2=3 \Rightarrow y_1^2-2y_1y_2+y_2^2=9$

$$\Rightarrow y_1^2+y_2^2-9 = 2y_1y_2 \quad \text{--- } (*)$$

又 M 為 A、B 中點 $\Rightarrow M\left(\underbrace{\frac{y_1^2+y_2^2}{8}}_{x}, \underbrace{\frac{y_1+y_2}{2}}_{y}\right)$

則由 $\begin{cases} y^2 = \frac{y_1^2+2y_1y_2+y_2^2}{4} \\ x = \frac{y_1^2+y_2^2}{8} \\ 2y_1y_2 = y_1^2+y_2^2-9 \end{cases} \Rightarrow \begin{cases} 4y^2 = 2(y_1^2+y_2^2)-9 \\ 8x = y_1^2+y_2^2 \end{cases}$

$$\Rightarrow 4y^2 = 16x-9$$

$$\Rightarrow y^2 = 4x - \frac{9}{4}$$