down $S_1'''A_0$ to S_0^{iv}, whereupon $P_0S_0^{iv}$, namely the ordinate of S_0^{iv}, is the desired fifth convergent to $\alpha = r_0$.

The two figures are to be imagined as superposed in such a way that the dotted line $R_5R_4R_3\ldots$ in Fig. 2 coincides with the heavy solid line in Fig. 1. The impossibility of drawing both lines on the same figure is indicated by the fact that $P_0R_0 = \sqrt{2} = 1{\cdot}41421^+$ and $P_0S_0^{iv} = 99/70 = 1{\cdot}41428^+$.

Just as the point S_0^{iv}, lying above R_0, has provided a fifth convergent to $\alpha = r_0$, namely $P_0S_0^{iv}$, so also the point S_1''', to the left of R_1, will provide a fourth convergent to r_1, the point S_2'' a third convergent to r_2, and the point S_3' a second convergent to r_3, where in all cases it is clear from the diagram that successive approximations are alternately by excess and by defect.

From the point of view of algebraic number theory, the most important property of these convergents is the fact that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \quad \text{for } n = 1, 2, 3, \ldots.$$

For $n = 1$, with $q_1 = a_1$, this property follows at once (see Fig. 1) from

$$\frac{p_1}{q_1} - \alpha = P_0S_0 - P_0R_0 = R_0S_0 = \frac{1}{a_1}.GE_1 < \frac{1}{a_1}.\frac{GE_1}{E_1R_1}$$
$$= \frac{1}{a_1}.\frac{A_1G}{A_0A_1} < \frac{1}{a_1^2}.$$

The reader is invited to determine how various other well known properties of convergents appear in geometrical form in the two diagrams.

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Symmetrical intersections of right circular cylinders

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The intersection of two circular cylinders of equal radius is not only of mathematical interest but also has application in both engineering and architecture. The joining of pipes of circular cross-section at a variety of given angles is an obvious example. The Romans and Normans, in using the barrel vault to span their buildings, were familiar with the geometry of intersecting cylinders where two such vaults crossed one another to form a cross vault. Larger numbers of equal intersecting cylinders arise in the following way.

When crystals grow they often do so by the accretion of material to form plane-faced polyhedra. If, however, the growing conditions are reversed (for example, by the increase of temperature or by the release of pressure) dissolution of the crystal may result; and a dissolved crystal may be expected to have rounded faces and curved edges, but vertices may still be comparatively sharp [1]. If one considers the dissolution of a symmetrical crystal one might imagine that under certain conditions each edge of the polyhedral form would be replaced by a piece of cylindrical surface, thus rounding the overall shape. As a first approximation one considers right circular cylinders, and as dissolution proceeds these small pieces of cylindrical surface will extend and eventually join up with other such surfaces from parallel edges to give the crystal a circular cross-section perpendicular to the direction of these edges. In this cylindrical approximation, material is considered to be removed from the crystal in directions perpendicular to the edge: in other words, the dissolution velocities in other directions are assumed to be so small that they may be reasonably neglected. The case of crystals belonging to the cubic system (for example, cubes and octahedra) is investigated here, as they are simpler to describe and are also of significance since they include many metals and semiconductors and also diamond. These resulting idealised shapes are of some interest. It is probably the first time that they have been considered; they bear a striking similarity to real cubic crystals where dissolution is known to have occurred; and although these mathematical solids are bounded by cylindrical surfaces their volumes, when referred to their diameters, have the remarkable property that they are independent of π.

The volume of the solid S_2 common to two equal right circular cylinders whose axes intersect at right angles was known to Archimedes [2, 3] and to Tsu Ch'ung-Chih [4]. They noted that this solid had square cross-sections which they compared in area with the circular sections of the inscribed sphere. The volume was found to be $\frac{2}{3}d^3$, where d is the diameter of each cylinder. This volume too is strangely simple, but the solid does not have the cubic symmetry required here.

The minimum number of cylinders needed for cubic symmetry is three, and their axes are mutually orthogonal. One might envisage a crystal cube dissolving, with preferential etching taking place at the twelve edges so that these become chamfered by cylindrical surfaces. Eventually these surfaces will join so that the remaining crystal would be just the intersection of the three cylinders whose axes are the three 4-fold symmetry axes of the cube. Fig. 1 shows a perspective view of the resulting shape S_3, wherein the axes of the cylinders are represented by the chain-dotted lines. Only two cylinders are drawn, for the third, if of equal length, would confuse the diagram. The horizontal lines which shade two of the facets of S_3 are generators of one of the cylinders. The solid is bounded by twelve such cylindrical facets, each having four edges and four vertices. The vertices are of two types: 3-fold where three facets meet, and 4-fold where four facets come together. The join of the two 3-fold vertices is a generator of the cylindrical surface and

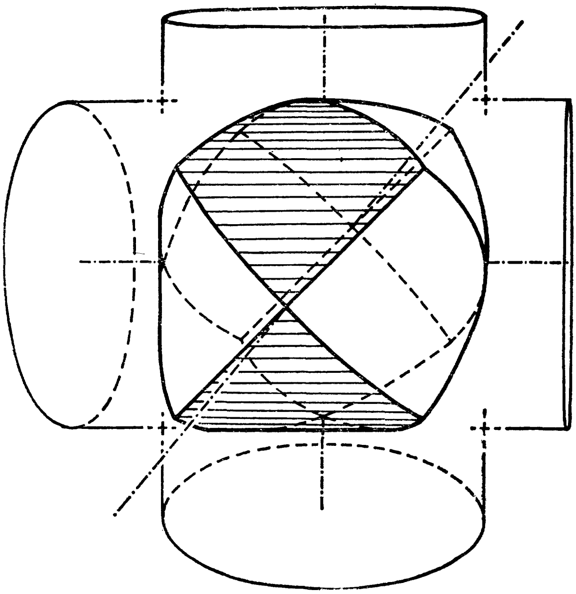

FIGURE 1.

thus lies wholly in the facet. The tangent plane touching the facet at this line is cut by other such tangent planes to form a rhombus and the solid thus bounded by all twelve tangent planes is a rhombic dodecahedron. Elementary integration (either by adding six caps to a cube, or by using cylindrical polar coordinates to evaluate the volume of a sector whose apex is the centre of S_3 and whose base is one cylindrical facet) gives the volume of S_3 as $(2-\sqrt{2})d^3$, where d is the diameter of the cylinders. The author has not yet discovered a neat way of finding the required volume by the elegant method of Archimedes for S_2, avoiding the use of calculus, but he has however checked the result by a simple Archimedean "Eureka" experiment!

If instead of the three 4-fold symmetry axes of a cube we take the four 3-fold axes (the body diagonals) for the axes of four symmetrically intersecting cylinders, we find that the solid S_4 common to all of them takes the shape depicted as a perspective view in Fig. 2. Again the axes have been drawn as chain-dotted lines. In the same way as S_3 was a cylindrical version of a rhombic dodecahedron, so S_4 is a kind of tetragonal trisoctahedron (or trapezoidal icositetrahedron) with twenty-four cylindrical facets. Some visible generators of one of the cylinders have been drawn in. Integration yields the volume of S_4: it is $\frac{3}{2}\sqrt{2}(2-\sqrt{3})d^3$.

The next simplest array of intersecting cylinders with cubic symmetry arises from the consideration of six cylinders of equal diameter whose axes are parallel to the face diagonals of a cube. The axes are 2-fold symmetry axes: the joins of mid-points of opposite edges of the cube. The shape of the resulting intersection solid S_6 is shown with these axes in Fig. 3. This has a total of thirty-six facets: twenty-four are of one type (roughly kite-shaped) and twelve are of another type (roughly rhombic). If tangent planes are drawn to these twelve pseudo-rhombic facets, touching them in the generators which join opposite vertices, a rhombic dodecahedron again results whose faces have their major diagonals along the generators mentioned.

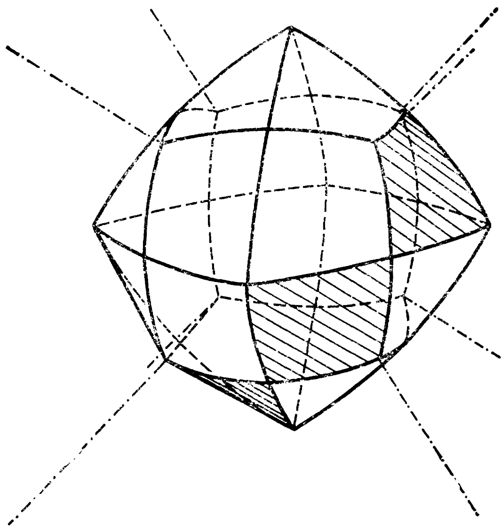

FIGURE 2.

The twenty-four kite-shaped facets approximate to those of another kind of tetragonal trisoctahedron. S_6 is thus a cylindrical analogue of the combination of the two polyhedra. Its volume is found by integration to be $\frac{2}{3}(3 + 2\sqrt{3} - 4\sqrt{2})d^3$.

The volumes of the intersection solids S_n become smaller as the number n of cylinders increases: $S_3 = 0{\cdot}586d^3$, $S_4 = 0{\cdot}568d^3$, $S_6 = 0{\cdot}538d^3$. The limiting case of a sphere, S_∞, has volume $\frac{1}{6}\pi d^3 = 0{\cdot}524d^3$. There are of course other ways of arranging several cylinders so that they intersect symmetrically, but they do not have quite the crystallographic importance of the three solids described here, which, despite the assumptions involved, do bear a resemblance to actual dissolution bodies. One may now deduce, from the shapes of these bodies, which are the most important crystallographic directions for dissolution in particular cases.

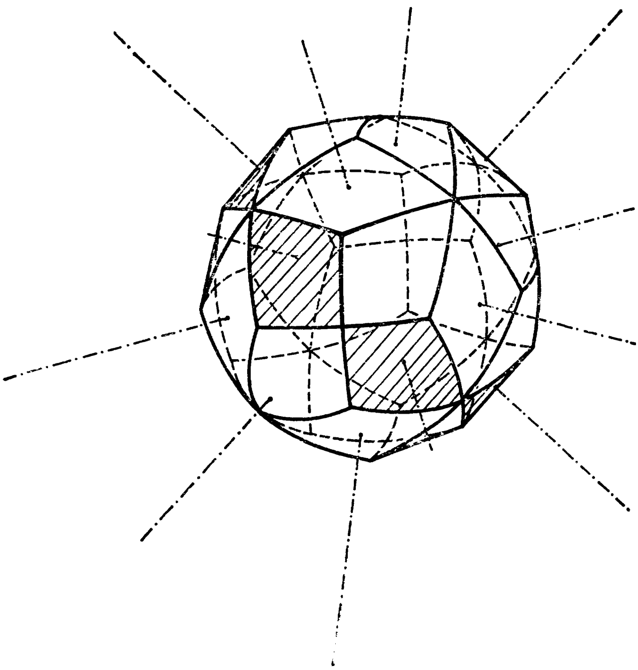

FIGURE 3.

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Gleanings far and near

The new hypnosis

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