1. (1) 定義 $\ln n = \int_1^n \frac{1}{x}dx$，證明： $\sum_{k=2}^{n} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$

(2) 利用定義，證明： $\ln ab = \ln a + \ln b$ (◊105新竹高中)

Solution:

(1) $\sum_{k=2}^{n} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$

Proof.

$\ln n = \int_1^n \frac{1}{x}dx = \int_1^2 \frac{1}{x}dx + \int_2^3 \frac{1}{x}dx + \ldots + \int_{n-1}^{n} \frac{1}{x}dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}dx$

設：$f(x) = \frac{1}{x}$ $(x > 0)$ 為單調遞減函數

若：$k < x < k+1, k \in \mathbb{Z}$

則：$\frac{1}{k+1} < \frac{1}{x} < \frac{1}{k}$

$\Longrightarrow \int_k^{k+1} \frac{1}{k+1}dx < \int_k^{k+1} \frac{1}{x}dx < \int_k^{k+1} \frac{1}{k}dx$

$\Longrightarrow \sum_{k=1}^{n-1} \frac{1}{k+1} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}dx < \sum_{k=1}^{n-1} \frac{1}{k}$

$\Longrightarrow \sum_{k=2}^{n} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$ □

(2) $\ln ab = \ln a + \ln b$

Proof.

$\ln ab = \int_1^{ab} \frac{1}{x}dx = \int_1^{a} \frac{1}{x}dx + \int_a^{ab} \frac{1}{x}dx = \int_1^{a} \frac{1}{x}dx + \int_1^{b} \frac{1}{x}dx = \ln a + \ln b$

claim: $\int_a^{ab} \frac{1}{x}dx = \int_1^{b} \frac{1}{x}dx$

利用變數變換，令$y = \frac{x}{a}, dy = \frac{1}{a}dx$ $\begin{pmatrix} x: a \to ab \\ y: 1 \to b \end{pmatrix}$

$\int_a^{ab} \frac{1}{x}dx = \int_1^{b} \frac{1}{ay}(ady) = \int_1^{b} \frac{1}{y}dy = \int_1^{b} \frac{1}{x}dx$

$\therefore \ln ab = \ln a + \ln b$ □