A Collection of Limits

March 28, 2011

Contents

Chapter 1

Short theoretical introduction

Consider a sequence of real numbers $(a_n)_{n\geq 1}$, and $l \in \overline{\mathbb{R}}$. We'll say that l represents the limit of $(a_n)_{n\geq 1}$ if any neighborhood of l contains all the terms of the sequence, starting from a certain index. We write this fact as $\lim\limits_{n\to\infty} a_n = l$, or $a_n \to l$.

We can rewrite the above definition into the following equivalence:

$$\lim_{n\to\infty} a_n = l \Leftrightarrow (\forall)V \in \mathcal{V}(l),\ (\exists)n_V \in \mathbb{N}^* \text{ such that } (\forall)n \geq n_V \Rightarrow a_n \in V.$$

One can easily observe from this definition that if a sequence is constant then it's limit is equal with the constant term.

We'll say that a sequence of real numbers $(a_n)_{n\geq 1}$ is convergent if it has limit and $\lim\limits_{n\to\infty} a_n \in \mathbb{R}$, or divergent if it doesn't have a limit or if it has the limit equal to $\pm\infty$.

Theorem: If a sequence has limit, then this limit is unique.

Proof: Consider a sequence $(a_n)_{n\geq 1} \subseteq \mathbb{R}$ which has two different limits $l', l'' \in \overline{\mathbb{R}}$. It follows that there exist two neighborhoods $V' \in \mathcal{V}(l')$ and $V'' \in \mathcal{V}(l'')$ such that $V' \cap V'' = \varnothing$. As $a_n \to l' \Rightarrow (\exists)n' \in \mathbb{N}^*$ such that $(\forall)n \geq n' \Rightarrow a_n \in V'$. Also, since $a_n \to l'' \Rightarrow (\exists)n'' \in \mathbb{N}^*$ such that $(\forall)n \geq n'' \Rightarrow a_n \in V''$. Hence $(\forall)n \geq \max\{n', n''\}$ we have $a_n \in V' \cap V'' = \varnothing$.

Theorem: Consider a sequence of real numbers $(a_n)_{n\geq 1}$. Then we have:

(i) $\lim\limits_{n\to\infty} a_n = l \in \mathbb{R} \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists)n_\varepsilon \in \mathbb{N}^*$ such that $(\forall)n \geq n_\varepsilon \Rightarrow |a_n - l| < \varepsilon$.

(ii) $\lim_{n\to\infty} a_n = \infty \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists)n_\varepsilon \in \mathbb{N}^*$ such that $(\forall)n \geq n_\varepsilon \Rightarrow a_n > \varepsilon$.

(iii) $\lim_{n\to\infty} a_n = -\infty \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists)n_\varepsilon \in \mathbb{N}^*$ such that $(\forall)n \geq n_\varepsilon \Rightarrow a_n < -\varepsilon$

Theorem: Let $(a_n)_{n\geq 1}$ a sequence of real numbers.

1. If $\lim_{n\to\infty} a_n = l$, then any subsequence of $(a_n)_{n\geq 1}$ has the limit equal to l.

2. If there exist two subsequences of $(a_n)_{n\geq 1}$ with different limits, then the sequence $(a_n)_{n\geq 1}$ is divergent.

3. If there exist two subsequences of $(a_n)_{n\geq 1}$ which cover it and have a common limit, then $\lim_{n\to\infty} a_n = l$.

Definition: A sequence $(x_n)_{n\geq 1}$ is a Cauchy sequence if $(\forall)\varepsilon > 0,\ (\exists)n_\varepsilon \in \mathbb{N}$ such that $|x_{n+p} - x_n| < \varepsilon,\ (\forall)n \geq n_\varepsilon,\ (\forall)p \in \mathbb{N}$.

Theorem: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Theorem: Any increasing and unbounded sequence has the limit ∞.

Theorem: Any increasing and bounded sequence converge to the upper bound of the sequence.

Theorem: Any convergent sequence is bounded.

Theorem(Cesaro lemma): Any bounded sequence of real numbers contains at least one convergent subsequence.

Theorem(Weierstrass theorem): Any monotonic and bounded sequence is convergent.

Theorem: Any monotonic sequence of real numbers has limit.

Theorem: Consider two convergent sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $a_n \leq b_n,\ (\forall)n \in \mathbb{N}^*$. Then we have $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Theorem: Consider a convergent sequence $(a_n)_{n\geq 1}$ and a real number a such that $a_n \leq a,\ (\forall)n \in \mathbb{N}^*$. Then $\lim_{n\to\infty} a_n \leq a$.

Theorem: Consider a convergent sequence $(a_n)_{n\geq 1}$ such that $\lim_{n\to\infty} a_n = a$. Them $\lim_{n\to\infty} |a_n| = |a|$.

Theorem: Consider two sequences of real numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $a_n \leq b_n,\ (\forall) n \in \mathbb{N}^*$. Then:

1. If $\lim\limits_{n\to\infty} a_n = \infty$ it follows that $\lim\limits_{n\to\infty} b_n = \infty$.

2. If $\lim\limits_{n\to\infty} b_n = -\infty$ it follows that $\lim\limits_{n\to\infty} a_n = -\infty$.

Limit operations:

Consider two sequences a_n and b_n which have limit. Then we have:

1. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$ (except the case $(\infty, -\infty)$).

2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$ (except the cases $(0, \pm\infty)$).

3. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$(except the cases $(0,0),\ (\pm\infty, \pm\infty)$).

4. $\lim\limits_{n\to\infty} a_n^{b_n} = (\lim\limits_{n\to\infty} a_n)^{\lim\limits_{n\to\infty} b_n}$ (except the cases $(1, \pm\infty),\ (\infty, 0),\ (0,0)$).

5. $\lim\limits_{n\to\infty} (\log_{a_n} b_n) = \log_{\lim\limits_{n\to\infty} a_n}(\lim\limits_{n\to\infty} b_n)$.

Trivial consequences:

1. $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$;

2. $\lim\limits_{n\to\infty} (\lambda a_n) = \lambda \lim\limits_{n\to\infty} a_n\ (\lambda \in \mathbb{R})$;

3. $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}\ (k \in \mathbb{N})$;

Theorem (Squeeze theorem): Let $(a_n)_{n\geq 1},\ (b_n)_{n\geq 1},\ (c_n)_{n\geq 1}$ be three sequences of real numbers such that $a_n \leq b_n \leq c_n$, $(\forall) n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = l \in \overline{\mathbb{R}}$. Then $\lim\limits_{n\to\infty} b_n = l$.

Theorem: Let $(x_n)_{n\geq 1}$ a sequence of real numbers such that $\lim\limits_{n\to\infty} (x_{n+1} - x_n) = \alpha \in \overline{\mathbb{R}}$.

1. If $\alpha > 0$, then $\lim\limits_{n\to\infty} x_n = \infty$.

2. If $\alpha < 0$, then $\lim\limits_{n\to\infty} x_n = -\infty$.

Theorem (Ratio test): Consider a sequence of real positive numbers $(a_n)_{n\geq 1}$, for which $l = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} \in \overline{\mathbb{R}}$.

1. If $l < 1$ then $\lim\limits_{n\to\infty} a_n = 0$.

2. If $l > 1$ then $\lim\limits_{n\to\infty} a_n = \infty$.

Proof: 1. Let $V = (\alpha, \beta) \in \mathcal{V}(l)$ with $l < \beta < 1$. Because $l = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$, there is some $n_0 \in \mathbb{N}^*$ such that $(\forall) n \geq n_0 \Rightarrow \dfrac{a_{n+1}}{a_n} \in V$, hence $(\forall) n \geq n_0 \Rightarrow \dfrac{a_{n+1}}{a_n} < 1$. That means starting from the index n_0 the sequence $(a_n)_{n\geq 1}$ is strictly decreasing. Since the sequence is strictly decreasing and it contains only positive terms, the sequence is bounded. Using Weierstrass Theorem, it follows that the sequence is convergent. We have:

$$a_{n+1} = \frac{a_{n+1}}{a_n} \cdot a_n \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \lim_{n\to\infty} a_n$$

which is equivalent with:

$$\lim_{n\to\infty} a_n(1 - l) = 0$$

which implies that $\lim\limits_{n\to\infty} a_n = 0$.

2. Denoting $b_n = \dfrac{1}{a_n}$ we have $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = \dfrac{1}{l} < 1$, hence $\lim\limits_{n\to\infty} b_n = 0$ which implies that $\lim\limits_{n\to\infty} a_n = \infty$.

Theorem: Consider a convergent sequence of real non-zero numbers $(x_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty} n \left(\dfrac{x_n}{x_{n-1}} - 1 \right) \in \mathbb{R}^*$. Then $\lim\limits_{n\to\infty} x_n = 0$.

Theorem(Cesaro-Stolz lemma): 1. Consider two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that:

(i) the sequence $(b_n)_{n\geq 1}$ is strictly increasing and unbounded;

(ii) the limit $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$ exists.

Then the sequence $\left(\dfrac{a_n}{b_n} \right)_{n\geq 1}$ is convergent and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$.

Proof: Let's consider the case $l \in \mathbb{R}$ and assume $(b_n)_{n\geq 1}$ is a strictly increasing sequence, hence $\lim\limits_{n\to\infty} b_n = \infty$. Now let $V \in \mathcal{V}(l)$, then there exists $\alpha > 0$ such that $(l-\alpha, l+\alpha) \subseteq V$. Let $\beta \in \mathbb{R}$ such that $0 < \beta < \alpha$. As $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$, there exists $k \in \mathbb{N}^*$ such that $(\forall) n \geq k \Rightarrow \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \in (l-\beta, l+\beta)$, which implies that:

$$(l-\beta)(b_{n+1}-b_n) < a_{n+1}-a_n < (l+\beta)(b_{n+1}-b_n),\ (\forall) n \geq k$$

Now writing this inequality from k to $n-1$ we have:

$$(l-\beta)(b_{k+1}-b_k) < a_{k+1}-a_k < (l+\beta)(b_{k+1}-b_k)$$

$$(l-\beta)(b_{k+2}-b_{k+1}) < a_{k+2}-a_{k+1} < (l+\beta)(b_{k+2}-b_{k+1})$$

$$\ldots$$

$$(l-\beta)(b_n-b_{n-1}) < a_n-a_{n-1} < (l+\beta)(b_n-b_{n-1})$$

Summing all these inequalities we find that:

$$(l-\beta)(b_n-b_k) < a_n-a_k < (l+\beta)(b_n-b_k)$$

As $\lim\limits_{n\to\infty} b_n = \infty$, starting from an index we have $b_n > 0$. The last inequality rewrites as:

$$(l-\beta)\left(1-\frac{b_k}{b_n}\right) < \frac{a_n}{b_n} - \frac{a_k}{b_n} < (l+\beta)\left(1-\frac{b_k}{b_n}\right) \Leftrightarrow$$

$$\Leftrightarrow (l-\beta) + \frac{a_k+(\beta-l)b_k}{b_n} < \frac{a_n}{b_n} < l+\beta+\frac{a_k-(\beta+l)b_k}{b_n}$$

As

$$\lim_{n\to\infty} \frac{a_k+(\beta-l)b_k}{b_n} = \lim_{n\to\infty} \frac{a_k-(\beta+l)b_k}{b_n} = 0$$

there exists an index $p \in \mathbb{N}^*$ such that $(\forall) n \geq p$ we have:

$$\frac{a_k+(\beta-l)b_k}{b_n},\ \frac{a_k-(\beta+l)b_k}{b_n} \in (\beta-\alpha, \alpha-\beta)$$

We shall look for the inequalities:

$$\frac{a_k+(\beta-l)b_k}{b_n} > \beta-\alpha$$

and

$$\frac{a_k-(\beta+l)b_k}{b_n} < \alpha-\beta$$

Choosing $m = \max\{k, p\}$, then $(\forall)n \geq m$ we have:

$$l - \alpha < \frac{a_n}{b_n} < l + \alpha$$

which means that $\frac{a_n}{b_n} \in V \Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$. It remains to prove the theorem when $l = \pm\infty$, but these cases can be proven analogous choosing $V = (\alpha, \infty)$ and $V = (-\infty, \alpha)$, respectively.

2. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ such that:

(i) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0,\ y_n \neq 0,\ (\forall)n \in \mathbb{N}^*$;

(ii) the sequence $(y_n)_{n\geq 1}$ is strictly decreasing;

(iii) the limit $\lim\limits_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = l \in \overline{\mathbb{R}}$.

Then the sequence $\left(\frac{x_n}{y_n}\right)_{n\geq 1}$ has a limit and $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = l$.

Remark: In problem's solutions we'll write directly $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$, and if the limit we arrive to belongs to $\overline{\mathbb{R}}$, then the application of Cesaro-Stolz lemma is valid.

Trivial consequences:

1. Consider a sequence $(a_n)_{n\geq 1}$ of strictly positive real numbers for which exists $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. Then we have:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Proof: Using Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty} (\ln \sqrt[n]{a_n}) = \lim_{n\to\infty} \frac{\ln a_n}{n} = \lim_{n\to\infty} \frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} = \lim_{n\to\infty} \ln\left(\frac{a_{n+1}}{a_n}\right) = \ln l$$

Then:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{a_n}} = e^{\lim\limits_{n\to\infty} (\ln \sqrt[n]{a_n})} = e^{\ln l} = l$$

2. Let $(x_n)_{n\geq 1}$ a sequence of real numbers which has limit. Then:

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n} = \lim_{n\to\infty} x_n$$

3. Let $(x_n)_{n\geq 1}$ a sequence of real positive numbers which has limit. Then:

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = \lim_{n\to\infty} x_n$$

Theorem (Reciprocal Cesaro-Stolz): Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ two sequences of real numbers such that:

(i) $(y_n)_{n\geq 1}$ is strictly increasing and unbounded;

(ii) the limit $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = l \in \overline{\mathbb{R}}$;

(iii) the limit $\lim\limits_{n\to\infty} \dfrac{y_n}{y_{n+1}} \in \mathbb{R}_+ \backslash \{1\}$.

Then the limit $\lim\limits_{n\to\infty} \dfrac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists and it is equal to l.

Theorem (exponential sequence): Let $a \in \mathbb{R}$. Consider the sequence $x_n = a^n,\ n \in \mathbb{N}^*$.

1. If $a \leq -1$, the sequence is divergent.

2. If $a \in (-1, 1)$, then $\lim\limits_{n\to\infty} x_n = 0$.

3. If $a = 1$, then $\lim\limits_{n\to\infty} x_n = 1$.

4. If $a > 1$, then $\lim\limits_{n\to\infty} x_n = \infty$.

Theorem (power sequence): Let $a \in \mathbb{R}$. Consider the sequence $x_n = n^a,\ n \in \mathbb{N}^*$.

1. If $a < 0$, then $\lim\limits_{n\to\infty} x_n = 0$.

2. If $a = 0$, then $\lim\limits_{n\to\infty} x_n = 1$.

3. If $a > 0$, then $\lim\limits_{n\to\infty} x_n = \infty$.

Theorem (polynomial sequence): Let $a_n = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0,\ (a_k \neq 0)$.

1. If $a_k > 0$, then $\lim\limits_{n\to\infty} a_n = \infty$.

2. If $a_k < 0$, then $\lim\limits_{n\to\infty} a_n = -\infty$.

Theorem: Let $b_n = \dfrac{a_k n^k + a_{k-1}n^{k-1} + \ldots + a_1 n + a_0}{b_p n^p + b_{p-1}n^{p-1} + \ldots + b_1 n + b_0}$, $(a_k \neq 0 \neq b_p)$.

1. If $k < p$, then $\lim\limits_{n\to\infty} b_n = 0$.

2. If $k = p$, then $\lim\limits_{n\to\infty} b_n = \dfrac{a_k}{b_p}$.

3. If $k > p$, then $\lim\limits_{n\to\infty} b_n = \dfrac{a_k}{b_p} \cdot \infty$.

Theorem: The sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$, $n \in \mathbb{N}^*$ is a strictly increasing and bounded sequence and $\lim\limits_{n\to\infty} a_n = e$.

Theorem: Consider a sequence $(a_n)_{n\geq 1}$ of real non-zero numbers such that $\lim\limits_{n\to\infty} a_n = 0$. Then $\lim\limits_{n\to\infty} (1 + a_n)^{\frac{1}{a_n}} = e$.

Proof: If $(b_n)_{n\geq 1}$ is a sequence of non-zero positive integers such that $\lim\limits_{n\to\infty} b_n = \infty$, we have $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{b_n}\right)^{b_n} = e$. Let $\varepsilon > 0$. From $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$, it follows that there exists $n'_\varepsilon \in \mathbb{N}^*$ such that $(\forall) n \geq n'_\varepsilon \Rightarrow \left| \left(1 + \dfrac{1}{n}\right)^n - e \right| < \varepsilon$. Also, since $\lim\limits_{n\to\infty} b_n = \infty$, there exists $n''_\varepsilon \in \mathbb{N}^*$ such that $(\forall) n \geq n''_\varepsilon \Rightarrow b_n > n'_\varepsilon$. Therefore there exists $n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\} \in \mathbb{N}^*$ such that $(\forall) n \geq n_\varepsilon \Rightarrow \left| \left(1 + \dfrac{1}{b_n}\right)^{b_n} - e \right| < \varepsilon$. This means that: $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{b_n}\right)^{b_n} = e$. The same property is fulfilled if $\lim\limits_{n\to\infty} b_n = -\infty$.

If $(c_n)_{n\geq 1}$ is a sequence of real numbers such that $\lim\limits_{n\to\infty} c_n = \infty$, then $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{c_n}\right)^{c_n} = e$. We can assume that $c_n > 1$, $(\forall) n \in \mathbb{N}^*$. Let's denote $d_n = \lfloor c_n \rfloor \in \mathbb{N}^*$. In this way $(d_n)_{n\geq 1}$ is sequence of positive integers with $\lim\limits_{n\to\infty} d_n = \infty$. We have:

$$d_n \leq c_n < d_n + 1 \Rightarrow \frac{1}{d_n + 1} < \frac{1}{c_n} \leq \frac{1}{d_n}$$

Hence it follows that:

$$\left(1 + \frac{1}{d_n + 1}\right)^d_n < \left(1 + \frac{1}{c_n}\right)^{d_n} \leq \left(1 + \frac{1}{c_n}\right)^{c_n} < \left(1 + \frac{1}{c_n}\right)^{d_n + 1} \leq \left(1 + \frac{1}{d_n}\right)^{d_n + 1}$$

Observe that:

$$\lim_{n\to\infty}\left(1+\frac{1}{d_n+1}\right)^{d_n} = \lim_{n\to\infty}\left(1+\frac{1}{d_n+1}\right)^{d_n+1}\cdot\left(1+\frac{1}{d_n+1}\right)^{-1} = e$$

and

$$\lim_{n\to\infty}\left(1+\frac{1}{d_n}\right)^{d_n+1} = \lim_{n\to\infty}\left(1+\frac{1}{d_n}\right)^{d_n}\cdot\left(1+\frac{1}{d_n}\right) = e$$

Using the Squeeze Theorem it follows that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{c_n}\right)^{c_n} = e$. The same property is fulfilled when $\lim\limits_{n\to\infty} c_n = -\infty$.

Now if the sequence $(a_n)_{n\geq 1}$ contains a finite number of positive or negative terms we can remove them and assume that the sequence contains only positive terms. Denoting $x_n = \dfrac{1}{a_n}$ we have $\lim\limits_{n\to\infty} x_n = \infty$. Then we have

$$\lim_{n\to\infty}(1+a_n)^{\frac{1}{a_n}} = \lim_{n\to\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e$$

If the sequence contains an infinite number of positive or negative terms, the same fact happens for the sequence $(x_n)_{n\geq 1}$ with $x_n = \dfrac{1}{a_n}$, $(\forall)n\in\mathbb{N}^*$. Let's denote by $(a'_n)_{n\geq 1}$ the subsequence of positive terms , and by $(a''_n)_{n\geq 1}$ the subsequence of negative terms. Also let $c'_n = \dfrac{1}{a'_n}$, $(\forall)n\in\mathbb{N}^*$ and $c''_n = \dfrac{1}{a''_n}$, $(\forall)n\in\mathbb{N}^*$. Then it follows that $\lim\limits_{n\to\infty} c'_n = \infty$ and $\lim\limits_{n\to\infty} c''_n = -\infty$. Hence:

$$\lim_{n\to\infty}(1+a'_n)^{\frac{1}{a'_n}} = \lim_{n\to\infty}\left(1+\frac{1}{c'_n}\right)^{c'_n} = e$$

and

$$\lim_{n\to\infty}(1+a''_n)^{\frac{1}{a''_n}} = \lim_{n\to\infty}\left(1+\frac{1}{c''_n}\right)^{c''_n} = e$$

Then it follows that: $\lim\limits_{n\to\infty}(1+a_n)^{\frac{1}{a_n}} = e$.

Consequence: Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ two sequences of real numbers such that $a_n \neq 1$, $(\forall)n\in\mathbb{N}^*$, $\lim\limits_{n\to\infty} a_n = 1$ and $\lim\limits_{n\to\infty} b_n = \infty$ or $\lim\limits_{n\to\infty} b_n = -\infty$. If there exists $\lim\limits_{n\to\infty}(a_n-1)b_n \in \overline{\mathbb{R}}$, then we have $\lim\limits_{n\to\infty} a_n^{b_n} = e^{\lim\limits_{n\to\infty}(a_n-1)b_n}$.

Theorem: Consider the sequence $(a_n)_{n\geq 0}$ defined by $a_n = \sum\limits_{k=0}^{n}\dfrac{1}{k!}$. We have $\lim\limits_{n\to\infty} a_n = e$.

Theorem: Let $(c_n)_{n\geq 1}$, a sequence defined by

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n,\ n \geq 1$$

Then $(c_n)_{n\geq 1}$ is strictly decreasing and bounded, and $\lim\limits_{n\to\infty} c_n = \gamma$, where γ is the Euler constant.

Recurrent sequences

A sequence $(x_n)_{n\geq 1}$ is a k-order recurrent sequence, if it is defined by a formula of the form

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, n_{n+k-1}),\ n \geq 1$$

with given $x_1, x_2, \ldots, x_k$. The recurrence is linear if f is a linear function. Second order recurrence formulas which are homogoeneus, with constant coefficients, have the form $x_{n+2} = \alpha x_{n+1} + \beta x_n,\ (\forall) n \geq 1$ with given x_1, x_2, α, β. To this recurrence formula we attach the equation $r^2 = \alpha r + \beta$, with r_1, r_2 as solutions.

If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then $x_n = Ar_1^n + Br_2^n$, where A, B are two real numbers, usually found from the terms x_1, x_2. If $r_1 = r_2 = r \in \mathbb{R}$, then $x_n = r^n(A + nB)$ and if $r_1, r_2 \in \mathbb{R}$, we have $r_1, r_2 = \rho(\cos\theta + i\sin\theta)$ so $x_n = \rho^n(\cos n\theta + i\sin n\theta)$.

Limit functions

Definition: Let $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}$) and $x_0 \in \overline{\mathbb{R}}$ and accumulation point of D. We'll say that $l \in \overline{\mathbb{R}}$ is the limit of the function f in x_0, and we write $\lim\limits_{x\to x_0} f(x) = l$, if for any neightborhood $\mathcal{V}$ of l, there is a neighborhood U of x_0, such that for any $x \in D \cap U \backslash \{x_0\}$, we have $f(x) \in \mathcal{V}$.

Theorem: Let $f : D \to \mathbb{R}$ ($D \subset \mathbb{R}$) and x_0 an accumulation point of D. Then $\lim\limits_{x\to x_0} f(x) = l$ ($l, x_0 \in \mathbb{R}$) if and only if $(\forall)\varepsilon > 0,\ (\exists)\delta_\varepsilon > 0,\ (\forall)x \in D\backslash\{x_0\}$ such that $|x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon$.

If $l = \pm\infty$, we have:

$\lim\limits_{x\to x_0} f(x) = \pm\infty \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists)\delta_\varepsilon > 0,\ (\forall)x \in D\backslash\{x_0\}$ such that $|x - x_0| < \delta_\varepsilon$, we have $f(x) > \varepsilon$ ($f(x) < \varepsilon$).

Theorem: Let $f : D \subset \mathbb{R} \Rightarrow \mathbb{R}$ and x_0 an accumulation point of D. Then $\lim\limits_{x\to x_0} f(x) = l$ ($l \in \overline{\mathbb{R}},\ x_0 \in \mathbb{R}$), if and only if $(\forall)(x_n)_{n\geq 1},\ x_n \in D\backslash\{x_0\},\ x_n \to x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = l$.

One-side limits

Definition: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$ an accumulation point of D. We'll say that $l_s \in \overline{\mathbb{R}}$ (or $l_d \in \overline{\mathbb{R}}$) is the left-side limit (or right-side limit) of f in x_0 if for any neigborhood $\mathcal{V}$ of l_s (or l_d), there is a neighborhood U of x_0, such that for any $x < x_0$, $x \in U \cap D \setminus \{x_0\}$ ($x > x_0$ respectively), $f(x) \in \mathcal{V}$.

We write $l_s = \lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0 - 0)$ and $l_d = \lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = f(x_0 + 0)$.

Theorem: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ an accumulation point of the sets $(-\infty, x_0) \cap D$ and $(x_0, \infty) \cap D$. Then f has the limit $l \in \overline{\mathbb{R}}$ if and only if f has equal one-side limits in x_0.

Remarkable limits

If $\lim\limits_{x \to x_0} f(x) = 0$, then:

1. $\lim\limits_{x \to x_0} \dfrac{\sin f(x)}{f(x)} = 1;$

2. $\lim\limits_{x \to x_0} \dfrac{\tan f(x)}{f(x)} = 1;$

3. $\lim\limits_{x \to x_0} \dfrac{\arcsin f(x)}{f(x)} = 1;$

4. $\lim\limits_{x \to x_0} \dfrac{\arctan f(x)}{f(x)} = 1;$

5. $\lim\limits_{x \to x_0} (1 + f(x))^{\frac{1}{f(x)}} = e$

6. $\lim\limits_{x \to x_0} \dfrac{\ln(1 + f(x))}{f(x)} = 1;$

7. $\lim\limits_{x \to x_0} \dfrac{a^{f(x)} - 1}{f(x)} = \ln a \ (a > 0);$

8. $\lim\limits_{x \to x_0} \dfrac{(1 + f(x))^r - 1}{f(x)} = r \ (r \in \mathbb{R});$

If $\lim\limits_{x \to x_0} f(x) = \infty$, then:

9. $\lim\limits_{x \to x_0} \left(1 + \dfrac{1}{f(x)}\right)^{f(x)} = e;$

10. $\lim\limits_{x \to x_0} \dfrac{\ln f(x)}{f(x)} = 0;$

Chapter 2

Problems

1. Evaluate:

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2+1}-\sqrt[3]{n^3-1}\right)$$

2. Evaluate:

$$\lim_{x\to-2}\frac{\sqrt[3]{5x+2}+2}{\sqrt{3x+10}-2}$$

3. Consider the sequence $(a_n)_{n\geq1}$, such that $\sum_{k=1}^{n} a_k = \frac{3n^2+9n}{2}$, $(\forall)n \geq 1$. Prove that this sequence is an arithmetical progression and evaluate:

$$\lim_{n\to\infty}\frac{1}{na_n}\sum_{k=1}^{n} a_k$$

4. Consider the sequence $(a_n)_{n\geq1}$ such that $a_1 = a_2 = 0$ and $a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^2 + b)$, where $0 \leq b \leq 1$. Prove that the sequence is convergent and evaluate $\lim\limits_{n\to\infty} a_n$.

5. Consider a sequence of real numbers $(x_n)_{n\geq1}$ such that $x_1 = 1$ and $x_n = 2x_{n-1} + \frac{1}{n}$, $(\forall)n \geq 2$. Evaluate $\lim\limits_{n\to\infty} x_n$.

6. Evaluate:

$$\lim_{n\to\infty}\left(n\left(\frac{4}{5}\right)^n + n^2\sin^n\frac{\pi}{6} + \cos\left(2n\pi + \frac{\pi}{n}\right)\right)$$

7. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k!\cdot k}{(n+1)!}$$

8. Evaluate:

$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)$$

9. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\frac{3^{3n}(n!)^3}{(3n)!}}$$

10. Consider a sequence of real positive numbers $(x_n)_{n\geq 1}$ such that $(n+1)x_{n+1}-nx_n<0$, $(\forall)n\geq 1$. Prove that this sequence is convergent and evaluate it's limit.

11. Find the real numbers a and b such that:

$$\lim_{n\to\infty}\left(\sqrt[3]{1-n^3}-an-b\right)=0$$

12. Let $p\in\mathbb{N}$ and $\alpha_1,\alpha_2,...,\alpha_p$ positive distinct real numbers. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\alpha_1^n+\alpha_2^n+\ldots+\alpha_p^n}$$

13. If $a\in\mathbb{R}^*$, evaluate:

$$\lim_{x\to -a}\frac{\cos x-\cos a}{x^2-a^2}$$

14. If $n\in\mathbb{N}^*$, evaluate:

$$\lim_{x\to 0}\frac{\ln(1+x+x^2+\ldots+x^n)}{nx}$$

15. Evaluate:

$$\lim_{n\to\infty}\left(n^2+n-\sum_{k=1}^{n}\frac{2k^3+8k^2+6k-1}{k^2+4k+3}\right)$$

16. Find $a\in\mathbb{R}^*$ such that:

$$\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

17. Evaluate:

$$\lim_{x\to 1}\frac{\sqrt[3]{x^2+7}-\sqrt{x+3}}{x^2-3x+2}$$

18. Evaluate:

$$\lim_{n\to\infty}\left(\sqrt{2n^2+n}-\lambda\sqrt{2n^2-n}\right)$$

where λ is a real number.

19. If $a, b, c \in \mathbb{R}$, evaluate:

$$\lim_{x\to\infty}\left(a\sqrt{x+1}+b\sqrt{x+2}+c\sqrt{x+3}\right)$$

20. Find the set $A \subset \mathbb{R}$ such that $ax^2+x+3 \geq 0,\ (\forall)a \in A, (\forall)x \in \mathbb{R}$. Then for any $a \in A$, evaluate:

$$\lim_{x\to\infty}\left(x+1-\sqrt{ax^2+x+3}\right)$$

21. If $k \in \mathbb{R}$, evaluate:

$$\lim_{n\to\infty} n^k\left(\sqrt{\frac{n}{n+1}}-\sqrt{\frac{n+2}{n+3}}\right)$$

22. If $k \in \mathbb{N}$ and $a \in \mathbb{R}_+\backslash\{1\}$, evaluate:

$$\lim_{n\to\infty} n^k(a^{\frac{1}{n}}-1)\left(\sqrt{\frac{n-1}{n}}-\sqrt{\frac{n+1}{n+2}}\right)$$

23. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$$

24. If $a > 0,\ p \geq 2$, evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt[p]{n^p+ka}}$$

25. Evaluate:

$$\lim_{n\to\infty}\frac{n!}{(1+1^2)(1+2^2)\cdot\ldots\cdot(1+n^2)}$$

26. Evaluate:

$$\lim_{n\to\infty}\left(\frac{2n^2-3}{2n^2-n+1}\right)^{\frac{n^2-1}{n}}$$

27. Evaluate:

$$\lim_{x\to 0}\frac{\sqrt{1+\sin^2 x}-\cos x}{1-\sqrt{1+\tan^2 x}}$$

28. Evaluate:

$$\lim_{x\to\infty}\left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right)^x$$

29. Evaluate:

$$\lim_{\substack{x\to 0\\ x>0}}(\cos x)^{\frac{1}{\sin x}}$$

30. Evaluate:

$$\lim_{x\to 0}(e^x+\sin x)^{\frac{1}{x}}$$

31. If $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$$

32. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ defined by:

$$a_n=\begin{cases} 1 & \text{if } n\leq k,\ k\in\mathbb{N}^* \\ \dfrac{(n+1)^k-n^k}{\binom{n}{k-1}} & \text{if } n>k \end{cases}$$

i)Evaluate $\lim\limits_{n\to\infty} a_n$.

ii)If $b_n = 1+\sum\limits_{k=1}^{n} k\cdot\lim\limits_{n\to\infty} a_n$, evaluate:

$$\lim_{n\to\infty}\left(\frac{b_n^2}{b_{n-1}b_{n+1}}\right)^n$$

33. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_{n+2} = \dfrac{x_{n+1}+x_n}{2}$, $(\forall)n \in \mathbb{N}^*$. If $x_1 \leq x_2$,

i)Prove that the sequence $(x_{2n+1})_{n\geq 0}$ is increasing, while the sequence $(x_{2n})_{n\geq 0}$ is decreasing;

ii)Prove that:

$$|x_{n+2}-x_{n+1}| = \frac{|x_2-x_1|}{2^n},\ (\forall)n\in\mathbb{N}^*$$

iii)Prove that:

$$2x_{n+2}+x_{n+1} = 2x_2+x_1,\ (\forall)n\in\mathbb{N}^*$$

iv)Prove that $(x_n)_{n\geq 1}$ is convergent and that it's limit is $\dfrac{x_1+2x_2}{3}$.

34. Let $a_n, b_n \in \mathbb{Q}$ such that $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$, $(\forall)n \in \mathbb{N}^*$. Evaluate $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$.

35. If $a > 0$, evaluate:

$$\lim_{x\to 0} \frac{(a+x)^x - 1}{x}$$

36. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ such that $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{a_n^2 - a_n + 1}{a_n}$. Prove that $(a_n)_{n\geq 1}$ is convergent and find it's limit.

37. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_0 \in (0,1)$ and $x_{n+1} = x_n - x_n^2 + x_n^3 - x_n^4$, $(\forall)n \geq 0$. Prove that this sequence is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

38. Let $a > 0$ and $b \in (a, 2a)$ and a sequence $x_0 = b$, $x_{n+1} = a + \sqrt{x_n(2a - x_n)}$, $(\forall)n \geq 0$. Study the convergence of the sequence $(x_n)_{n\geq 0}$.

39. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n+1} \arctan \frac{1}{2k^2}$$

40. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{4k^4+1}$$

41. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1+3+3^2+\ldots+3^k}{5^{k+2}}$$

42. Evaluate:

$$\lim_{n\to\infty} \left(n+1 - \sum_{i=2}^{n} \sum_{k=2}^{i} \frac{k-1}{k!} \right)$$

43. Evaluate:

$$\lim_{n\to\infty} \frac{1^1 + 2^2 + 3^3 + \ldots + n^n}{n^n}$$

44. Consider the sequence $(a_n)_{n\geq 1}$ such that $a_0 = 2$ and $a_{n-1} - a_n = \frac{n}{(n+1)!}$. Evaluate $\lim\limits_{n\to\infty} ((n+1)! \ln a_n)$.

45. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ with $x_1 = a > 0$ and $x_{n+1} = \dfrac{x_1 + 2x_2 + 3x_3 + \ldots + nx_n}{n}$, $n \in \mathbb{N}^*$. Evaluate it's limit.

46. Using $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{k^2} = \dfrac{\pi^2}{6}$, evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$$

47. Consider the sequence $(x_n)_{n\geq 1}$ defined by $x_1 = a$, $x_2 = b$, $a < b$ and $x_n = \dfrac{x_{n-1} + \lambda x_{n-2}}{1+\lambda}$, $n \geq 3$, $\lambda > 0$. Prove that this sequence is convergent and find it's limit.

48. Evaluate:

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$$

49. Consider the sequence $(x_n)_{n\geq 1}$ defined by $x_1 = 1$ and $x_n = \dfrac{1}{1 + x_{n-1}}$, $n \geq 2$. Prove that this sequence is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

50. If $a, b \in \mathbb{R}^*$, evaluate:

$$\lim_{x\to 0} \frac{\ln(\cos ax)}{\ln(\cos bx)}$$

51. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \{x\} & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim\limits_{x\to\alpha} f(x)$ exists.

52. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim\limits_{x\to\alpha} f(x)$ exists.

53. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that $x_1 > 0$ and $3x_n = 2x_{n-1} + \dfrac{a}{x_{n-1}^2}$, where a is a real positive number. Prove that x_n is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

54. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ such that $a_1 = 12$ and $a_{n+1} = a_n\left(1 + \dfrac{3}{n+1}\right)$. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{a_k}$$

55. Evaluate:

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+1}}\right)^n$$

56. If $a \in \mathbb{R}$, evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\lfloor k^2 a\rfloor}{n^3}$$

57. Evaluate:

$$\lim_{n\to\infty} 2^n\left(\sum_{k=1}^{n}\frac{1}{k(k+2)}-\frac{1}{4}\right)^n$$

58. Consider the sequence $(a_n)_{n\geq 1}$, such that $a_n > 0,\ (\forall)n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} n(a_{n+1} - a_n) = 1$. Evaluate $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$.

59. Evaluate:

$$\lim_{n\to\infty}\frac{1+2\sqrt{2}+3\sqrt{3}+\ldots+n\sqrt{n}}{n^2\sqrt{n}}$$

60. Evaluate:

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{2x-\pi}}$$

61. Evaluate:

$$\lim_{n\to\infty} n^2\ln\left(\cos\frac{1}{n}\right)$$

62. Given $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$$

63. Let $\alpha > \beta > 0$ and the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

i)Prove that $(\exists)(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \mathbb{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^n = x_n A + y_n B,\ (\forall)n \geq 1$$

ii)Evaluate $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}$.

64. If $a \in \mathbb{R}$ such that $|a| < 1$ and $p \in \mathbb{N}^*$ is given, evaluate:

$$\lim_{n\to\infty} n^p \cdot a^n$$

65. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$$

66. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \to 1 \\ x<1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}}$$

67. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \to 1 \\ x<1}} \frac{1-\cos(n \arccos x)}{1-x^2}$$

68. Study the convergence of the sequence:

$$x_{n+1} = \frac{x_n + a}{x_n + 1},\ n \geq 1,\ x_1 \geq 0,\ a > 0$$

69. Consider two sequences of real numbers $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ such that $x_0 = y_0 = 3,\ x_n = 2x_{n-1} + y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1},\ (\forall)n \geq 1$. Evaluate $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$.

70. Evaluate:

$$\lim_{x\to 0} \frac{\tan x - x}{x^2}$$

71. Evaluate:

$$\lim_{x\to 0} \frac{\tan x - \arctan x}{x^2}$$

72. Let $a > 0$ and a sequence of real numbers $(x_n)_{n\geq 0}$ such that $x_n \in (0, a)$ and $x_{n+1}(a - x_n) > \dfrac{a^2}{4},\ (\forall)n \in \mathbb{N}$. Prove that $(x_n)_{n\geq 1}$ is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

73. Evaluate:

$$\lim_{n\to\infty} \cos\left(n\pi \sqrt[2n]{e}\right)$$

74. Evaluate:

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\tan \frac{(n-1)\pi}{2n}}$$

75. Evaluate:

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} \binom{n}{k}}$$

76. If $a > 0$, evaluate:

$$\lim_{n\to\infty} \frac{a + \sqrt{a} + \sqrt[3]{a} + \ldots + \sqrt[n]{a} - n}{\ln n}$$

77. Evaluate:

$$\lim_{n\to\infty} n \ln \tan\left(\frac{\pi}{4} + \frac{\pi}{n}\right)$$

78. Let $k \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_k \in \mathbb{R}$ such that $a_0 + a_1 + a_2 + \ldots + a_k = 0$. Evaluate:

$$\lim_{n\to\infty} \left(a_0 \sqrt[3]{n} + a_1 \sqrt[3]{n+1} + \ldots + a_k \sqrt[3]{n+k}\right)$$

79. Evaluate:

$$\lim_{n\to\infty} \sin\left(n\pi \sqrt[3]{n^3 + 3n^2 + 4n - 5}\right)$$

80. Evaluate:

$$\lim_{\substack{x \to 1 \\ x<1}} \frac{2 \arcsin x - \pi}{\sin \pi x}$$

81. Evaluate:

$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k \ln k}$$

82. Evaluate:

$$\lim_{n\to\infty} \left[\lim_{x\to 0} \left(1 + \sum_{k=1}^{n} \sin^2(kx)\right)^{\frac{1}{n^3 x^2}}\right]$$

83. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(k+1)(k+2) \cdot \ldots \cdot (k+p)}{n^{p+1}}$$

84. If $\alpha_n \in \left(0, \frac{\pi}{4}\right)$ is a root of the equation $\tan\alpha + \cot\alpha = n,\ n \geq 2$, evaluate:

$$\lim_{n\to\infty} (\sin\alpha_n + \cos\alpha_n)^n$$

85. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{\binom{n+k}{2}}}{n^2}$$

86. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)}$$

87. Evaluate:

$$\lim_{x\to 0}\frac{\arctan x-\arcsin x}{x^3}$$

88. If $\alpha > 0$, evaluate:

$$\lim_{n\to\infty}\frac{(n+1)^\alpha-n^\alpha}{n^{\alpha-1}}$$

89. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{2^k}$$

90. Evaluate:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(k+1)(k+2)}{2^k}$$

91. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_1 \in (0,1)$ and $x_{n+1} = x_n^2 - x_n + 1,\ (\forall) n \in \mathbb{N}$. Evaluate:

$$\lim_{n\to\infty}(x_1 x_2 \cdot \ldots \cdot x_n)$$

92. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x\to 0}\frac{1-\cos x\cdot\cos 2x\cdot\ldots\cdot\cos nx}{x^2}$$

93. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that x_n is the real root of the equation $x^3+nx-n=0,\ n\in\mathbb{N}^*$. Prove that this sequence is convergent and find it's limit.

94. Evaluate:

$$\lim_{x\to 2}\frac{\arctan x-\arctan 2}{\tan x-\tan 2}$$

95. Evaluate:

$$\lim_{n\to\infty} \frac{1+\sqrt[2^2]{2!}+\sqrt[3^2]{3!}+\ldots+\sqrt[n^2]{n!}}{n}$$

96. Let $(x_n)_{n\geq 1}$ such that $x_1 > 0$, $x_1 + x_1^2 < 1$ and $x_{n+1} = x_n + \frac{x_n^2}{n^2}$, $(\forall) n \geq 1$. Prove that the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 2}$, $y_n = \frac{1}{x_n} - \frac{1}{n-1}$ are convergent.

97. Evaluate:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin \frac{2i}{n^2}$$

98. If $a > 0,\ a \neq 1$, evaluate:

$$\lim_{x\to a} \frac{x^x - a^x}{a^x - a^a}$$

99. Consider a sequence of positive real numbers $(a_n)_{n\geq 1}$ such that $a_{n+1} - \frac{1}{a_{n+1}} = a_n + \frac{1}{a_n}$, $(\forall) n \geq 1$. Evaluate:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right)$$

100. Evaluate:

$$\lim_{x\to 0} \frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}}$$

Chapter 3

Solutions

1. Evaluate:

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2+1}-\sqrt[3]{n^3-1}\right)$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2+1}-\sqrt[3]{n^3-1}\right) &= \lim_{n\to\infty}\frac{n^3+2n^2+1-n^3+1}{\sqrt[3]{(n^3+2n^2+1)^2}+\sqrt[3]{(n^3-1)(n^3+2n^2+1)}+\sqrt[3]{(n^3-1)^2}}\\ &= \lim_{n\to\infty}\frac{n^2\left(2+\frac{2}{n}\right)}{n^2\left[\sqrt[3]{\left(1+\frac{2}{n}+\frac{1}{n^3}\right)^2}+\sqrt[3]{\left(1-\frac{1}{n^3}\right)\left(1+\frac{2}{n}+\frac{1}{n^3}\right)}+\sqrt[3]{\left(1-\frac{1}{n^3}\right)^2}\right]}\\ &= \frac{2}{3}\end{aligned}$$

2. Evaluate:

$$\lim_{x\to-2}\frac{\sqrt[3]{5x+2}+2}{\sqrt{3x+10}-2}$$

Solution:

$$\begin{aligned}\lim_{x\to-2}\frac{\sqrt[3]{5x+2}+2}{\sqrt{3x+10}-2} &= \lim_{x\to-2}\frac{\frac{5x+10}{\sqrt[3]{(5x+2)^2}-2\sqrt[3]{5x+2}+4}}{\frac{3x+6}{\sqrt{3x+10}+2}}\\ &= \frac{5}{3}\lim_{x\to-2}\frac{\sqrt{3x+10}+2}{\sqrt[3]{(5x+2)^2}-2\sqrt[3]{5x+2}+4}\\ &= \frac{5}{9}\end{aligned}$$

3. Consider the sequence $(a_n)_{n\geq 1}$, such that $\displaystyle\sum_{k=1}^{n} a_k = \frac{3n^2+9n}{2}$, $(\forall)n \geq 1$.
Prove that this sequence is an arithmetical progression and evaluate:

$$\lim_{n\to\infty} \frac{1}{na_n} \sum_{k=1}^{n} a_k$$

Solution: For $n = 1$ we get $a_1 = 6$. Then $a_1 + a_2 = 15$, so $a_2 = 9$ and the ratio is $r = 3$. Therefore the general term is $a_n = 6 + 3(n-1) = 3(n+1)$. So:

$$\lim_{n\to\infty} \frac{1}{na_n} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{n+3}{2n+2} = \frac{1}{2}$$

4. Consider the sequence $(a_n)_{n\geq 1}$ such that $a_1 = a_2 = 0$ and $a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^2 + b)$, where $0 \leq b < 1$. Prove that the sequence is convergent and evaluate $\lim_{n\to\infty} a_n$.

Solution: We have $a_2 - a_1 = 0$ and $a_3 - a_2 = \frac{b}{3} \geq 0$, so assuming $a_{n-1} \geq a_{n-2}$ and $a_n \geq a_{n-1}$, we need to show that $a_{n+1} \geq a_n$. The recurrence equation gives us:

$$a_{n+1} - a_n = \frac{1}{3}(a_n - a_{n-1} + a_{n-1}^2 - a_{n-2}^2)$$

Therefore it follows that the sequence is monotonically increasing. Also, because $b \leq 1$, we have $a_3 = \frac{b}{3} < 1$, $a_4 = \frac{4b}{9} < 1$. Assuming that $a_{n-1},\ a_n < 1$, it follows that:

$$a_{n+1} = \frac{1}{3}(b + a_n + a_{n-1}^2) < \frac{1}{3}(1+1+1) = 1$$

Hence $a_n \in [0,1)$, $(\forall)n \in \mathbb{N}^*$, which means the sequence is bounded. From Weierstrass theorem it follows that the sequence is convergent. Let then $\lim_{n\to\infty} a_n = l$. By passing to limit in the recurrence relation, we have:

$$l^2 - 2l + b = 0 \Leftrightarrow (l-1)^2 = 1 - b \Rightarrow l = 1 \pm \sqrt{1-b}$$

Because $1 + \sqrt{1-b} > 1$ and $a_n \in [0,1)$, it follows that $\lim_{n\to\infty} a_n = 1 - \sqrt{1-b}$.

5. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_1 = 1$ and $x_n = 2x_{n-1} + \frac{1}{2}$, $(\forall)n \geq 2$. Evaluate $\lim_{n\to\infty} x_n$.

Solution: Let's evaluate a few terms:

$$x_2 = 2 + \frac{1}{2}$$

$$x_3 = 2^2 + 2 \cdot \frac{1}{2} + \frac{1}{2} = 2^2 + \frac{1}{2}(2^2 - 1)$$

$$x_4 = 2^3 + 2^2 - 1 + \frac{1}{2} = 2^3 + \frac{1}{2}(2^3 - 1)$$

$$x_5 = 2^4 + 2^3 - 1 + \frac{1}{2} = 2^4 + \frac{1}{2}(2^4 - 1)$$

and by induction we can show immediately that $x_n = 2^{n-1} + \frac{1}{2}(2^{n-1} - 1)$. Thus $\lim\limits_{n\to\infty} x_n = \infty$.

6. Evaluate:

$$\lim_{n\to\infty}\left(n\left(\frac{4}{5}\right)^n + n^2\sin^n\frac{\pi}{6} + \cos\left(2n\pi + \frac{\pi}{n}\right)\right)$$

Solution: We have:

$$\lim_{n\to\infty}\frac{\frac{4^{n+1}\cdot(n+1)}{5^{n+1}}}{\frac{4^n\cdot n}{5^n}} = \lim_{n\to\infty}\frac{4(n+1)}{5n} = \frac{4}{5} < 1$$

Thus using the ratio test it follows that $\lim\limits_{n\to\infty} n\left(\frac{4}{5}\right)^n = 0$. Also

$$\lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty}\frac{n^2+2n+1}{2n^2} = \frac{1}{2} < 1$$

From the ratio test it follows that $\lim\limits_{n\to\infty}\frac{n^2}{2^n} = \lim\limits_{n\to\infty} n^2\sin^n\frac{\pi}{6} = 0$. Therefore the limit is equal to

$$\lim_{n\to\infty}\cos\left(2n\pi + \frac{\pi}{n}\right) = \lim_{n\to\infty}\cos\frac{\pi}{n} = \cos 0 = 1$$

7. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k!\cdot k}{(n+1)!}$$

Solution:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k!\cdot k}{(n+1)!} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)!-k!}{(n+1)!} = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1$$

8. Evaluate:

$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) &= \lim_{n\to\infty}\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)\\
&= \lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right)\\
&= \lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{(r-1)(r+1)}{r^2}\right)\\
&= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\\
&= \frac{1}{2}
\end{aligned}$$

9. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\frac{3^{3n}(n!)^3}{(3n)!}}$$

Solution: Define $a_n = \dfrac{3^{3n}(n!)^3}{(3n)!}$. Then:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\\
&= \lim_{n\to\infty}\frac{3^{3n+3}[(n+1)!]^3}{(3n+3)!}\cdot\frac{(3n)!}{3^{3n}(n!)^3}\\
&= \lim_{n\to\infty}\frac{27(n+1)^3}{(3n+1)(3n+2)(3n+3)}\\
&= 1
\end{aligned}$$

10. Consider a sequence of real positive numbers $(x_n)_{n\geq 1}$ such that $(n+1)x_{n+1}-nx_n<0$, $(\forall)n\geq 1$. Prove that this sequence is convergent and evaluate it's limit.

Solution: Because $nx_n > (n+1)x_{n+1}$, we deduce that $x_1 > 2x_2 > 3x_3 > \ldots > nx_n$, whence $0 < x_n < \dfrac{x_1}{n}$. Using the Squeeze Theorem it follows that $\lim\limits_{n\to\infty} x_n = 0$.

11. Find the real numbers a and b such that:

$$\lim_{n\to\infty}\left(\sqrt[3]{1-n^3}-an-b\right)=0$$

Solution: We have:

$$\begin{aligned}
b &= \lim_{n\to\infty}\left(\sqrt[3]{1-n^3}-an\right)\\
&= \lim_{n\to\infty}\frac{1-n^3-a^3n^3}{\sqrt[3]{(1-n^3)^2}+\sqrt[3]{an(1-n^3)}+\sqrt[3]{a^2n^2}}\\
&= \lim_{n\to\infty}\frac{n\left(-1-a^3+\frac{1}{n^3}\right)}{\sqrt[3]{\left(\frac{1}{n^3}-1\right)^2}+\sqrt[3]{a\left(\frac{1}{n^5}-\frac{1}{n^2}\right)}+\sqrt[3]{\frac{a^2}{n^4}}}
\end{aligned}$$

If $-1-a^3\neq 0$, it follows that $b=\pm\infty$, which is false. Hence $a^3=-1\Rightarrow a=-1$ and so $b=0$.

12. Let $p\in\mathbb{N}$ and $\alpha_1,\alpha_2,...,\alpha_p$ positive distinct real numbers. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\alpha_1^n+\alpha_2^n+\ldots+\alpha_p^n}$$

Solution: WLOG let $\alpha_j=\max\{\alpha_1,\alpha_2,\ldots,\alpha_p\},\ 1\le j\le p$. Then:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\alpha_1^n+\alpha_2^n+\ldots+\alpha_p^n} &= \lim_{n\to\infty}\alpha_j\sqrt[n]{\left(\frac{\alpha_1}{\alpha_j}\right)^n+\left(\frac{\alpha_2}{\alpha_j}\right)^n+\ldots+\left(\frac{\alpha_{j-1}}{\alpha_j}\right)^n+1+\left(\frac{\alpha_{j+1}}{\alpha_j}\right)^n+\ldots+\left(\frac{\alpha_p}{\alpha_j}\right)^n}\\
&= \alpha_j\\
&= \max\{\alpha_1,\alpha_2,\ldots,\alpha_p\}
\end{aligned}$$

13. If $a\in\mathbb{R}^*$, evaluate:

$$\lim_{x\to -a}\frac{\cos x-\cos a}{x^2-a^2}$$

Solution:

$$\begin{aligned}
\lim_{x\to -a}\frac{\cos x-\cos a}{x^2-a^2} &= \lim_{x\to -a}\frac{-2\sin\frac{x+a}{2}\cdot\sin\frac{x-a}{2}}{(x-a)(x+a)}\\
&= \lim_{x\to -a}\frac{\sin\frac{x+a}{2}}{\frac{x+a}{2}}\cdot\lim_{x\to -a}\frac{\sin\frac{x-a}{2}}{a-x}\\
&= \lim_{x\to -a}\frac{\sin\frac{x-a}{2}}{a-x}\\
&= -\frac{\sin a}{2a}
\end{aligned}$$

14. If $n\in\mathbb{N}^*$, evaluate:

$$\lim_{x\to 0}\frac{\ln(1+x+x^2+\ldots+x^n)}{nx}$$

Solution: Using $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$, we have:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+x+x^2+\ldots+x^n)}{nx} &= \lim_{x\to 0}\frac{\ln(1+x+x^2+\ldots+x^n)}{x+x^2+\ldots+x^n}\cdot\lim_{x\to 0}\frac{x+x^2+\ldots+x^n}{nx}\\ &= \lim_{x\to 0}\frac{x+x^2+\ldots+x^n}{nx}\\ &= \lim_{x\to 0}\frac{1+x+\ldots+x^{n-1}}{n}\\ &= \frac{1}{n}\end{aligned}$$

15. Evaluate:

$$\lim_{n\to\infty}\left(n^2+n-\sum_{k=1}^{n}\frac{2k^3+8k^2+6k-1}{k^2+4k+3}\right)$$

Solution: Telescoping, we have:

$$\begin{aligned}\lim_{n\to\infty}\left(n^2+n-\sum_{k=1}^{n}\frac{2k^3+8k^2+6k-1}{k^2+4k+3}\right) &= \lim_{n\to\infty}\left(n^2+n-2\sum_{k=1}^{n}k+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+3}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+1}-\sum_{k=1}^{n}\frac{1}{k+3}\right)\\ &= \frac{5}{12}-\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{n+2}+\frac{1}{n+3}\right)\\ &= \frac{5}{12}\end{aligned}$$

16. Find $a\in\mathbb{R}^*$ such that:

$$\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

Solution: Observe that:

$$\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{4}\lim_{x\to 0}\frac{2\sin^2\frac{ax}{2}}{\frac{a^2x^2}{4}}=\frac{a^2}{2}$$

and

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x}=\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi-x}=1$$

Therefore $\dfrac{a^2}{2}=1$, which implies $a=\pm\sqrt{2}$.

17. Evaluate:

$$\lim_{x \to 1} \frac{\sqrt[3]{x^2+7} - \sqrt{x+3}}{x^2-3x+2}$$

Solution:

$$\begin{aligned}
\lim_{x \to 1} \frac{\sqrt[3]{x^2+7} - \sqrt{x+3}}{x^2-3x+2} &= \lim_{x \to 1} \frac{\sqrt[3]{x^2+7} - 3}{x^2-3x+2} + \lim_{x \to 1} \frac{2 - \sqrt{x+3}}{x^2-3x+2} \\
&= \lim_{x \to 1} \frac{x+1}{(x-2)\left(\sqrt[3]{(x^2+7)^2} + 2\sqrt[3]{x^2+7} + 4\right)} + \lim_{x \to 1} \frac{1}{(2-x)(2+\sqrt{x+3})} \\
&= -\frac{2}{12} + \frac{1}{4} \\
&= \frac{1}{12}
\end{aligned}$$

18. Evaluate:

$$\lim_{n \to \infty} \left(\sqrt{2n^2+n} - \lambda\sqrt{2n^2-n}\right)$$

where λ is a real number.

Solution:

$$\begin{aligned}
\lim_{n \to \infty} \left(\sqrt{2n^2+n} - \lambda\sqrt{2n^2-n}\right) &= \lim_{n \to \infty} \frac{2n^2+n-\lambda^2\left(2n^2-n\right)}{\sqrt{2n^2+n}+\lambda\sqrt{2n^2-n}} \\
&= \lim_{n \to \infty} \frac{2n^2\left(1-\lambda^2\right)+n\left(1+\lambda^2\right)}{n\left(\sqrt{2+\frac{1}{n}}+\lambda\sqrt{2-\frac{1}{n}}\right)} \\
&= \lim_{n \to \infty} \frac{2n\left(1-\lambda^2\right)+\left(1+\lambda^2\right)}{\sqrt{2+\frac{1}{n}}+\lambda\sqrt{2-\frac{1}{n}}} \\
&= \begin{cases} +\infty & \text{if } \lambda \in (-\infty, 1) \\ \dfrac{\sqrt{2}}{2} & \text{if } \lambda = 1 \\ -\infty & \text{if } \lambda \in (1, +\infty) \end{cases}
\end{aligned}$$

19. If $a, b, c \in \mathbb{R}$, evaluate:

$$\lim_{x \to \infty} \left(a\sqrt{x+1} + b\sqrt{x+2} + c\sqrt{x+3}\right)$$

Solution: If $a+b+c \neq 0$, we have:

$$\lim_{x\to\infty}\left(a\sqrt{x+1}+b\sqrt{x+2}+c\sqrt{x+3}\right)=\lim_{x\to\infty}\sqrt{x}\left(a\sqrt{1+\frac{1}{x}}+b\sqrt{1+\frac{2}{x}}+c\sqrt{1+\frac{3}{x}}\right)$$
$$=\lim_{x\to\infty}\sqrt{x}\,(a+b+c)$$
$$=\begin{cases}-\infty & \text{if } a+b+c<0\\ \infty & \text{if } a+b+c>0\end{cases}$$

If $a+b+c=0$, then:

$$\lim_{x\to\infty}\left(a\sqrt{x+1}+b\sqrt{x+2}+c\sqrt{x+3}\right)=\lim_{x\to\infty}\left(a\sqrt{x+1}-a+b\sqrt{x+2}-b+c\sqrt{x+3}-c\right)$$
$$=\lim_{x\to\infty}\left(\frac{a}{\sqrt{\frac{1}{x}+\frac{1}{x^2}}+\frac{1}{x}}+\frac{b+\frac{b}{x}}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\frac{1}{x}}+\frac{c+\frac{2c}{x}}{\sqrt{\frac{1}{x}+\frac{3}{x^2}}+\frac{1}{x}}\right)$$
$$=0$$

20. Find the set $A\subset\mathbb{R}$ such that $ax^2+x+3\geq 0,\ (\forall)a\in A,(\forall)x\in\mathbb{R}$. Then for any $a\in A$, evaluate:

$$\lim_{x\to\infty}\left(x+1-\sqrt{ax^2+x+3}\right)$$

Solution: We have $ax^2+x+3\geq 0,\ (\forall)x\in\mathbb{R}$ if $a>0$ and $\Delta_x\leq 0$, whence $a\in\left[\frac{1}{12},\infty\right)$. Then:

$$\lim_{x\to\infty}\left(x+1-\sqrt{ax^2+x+3}\right)=\lim_{x\to\infty}\frac{(1-a)x^2+x-2}{x+1+\sqrt{ax^2+x+3}}$$
$$=\lim_{x\to\infty}\frac{(1-a)x+1-\frac{2}{x}}{1+\frac{1}{x}+\sqrt{a+\frac{1}{x}+\frac{3}{x^2}}}$$
$$=\begin{cases}\infty & \text{if } a\in\left[\frac{1}{12},1\right)\\ \frac{1}{2} & \text{if } a=1\\ -\infty & \text{if } a\in(1,\infty)\end{cases}$$

21. If $k\in\mathbb{R}$, evaluate:

$$\lim_{n\to\infty}n^k\left(\sqrt{\frac{n}{n+1}}-\sqrt{\frac{n+2}{n+3}}\right)$$

Solution:

$$\lim_{n\to\infty} n^k\left(\sqrt{\frac{n}{n+1}}-\sqrt{\frac{n+2}{n+3}}\right) = \lim_{n\to\infty}\frac{n^k}{(n+1)(n+2)}\cdot\lim_{n\to\infty}\frac{-2}{\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n+2}{n+3}}}$$
$$= \lim_{n\to\infty}\frac{-n^k}{(n+1)(n+2)}$$
$$= \begin{cases} 0 & \text{if } k<2 \\ -1 & \text{if } k=2 \\ -\infty & \text{if } k>2 \end{cases}$$

22. If $k \in \mathbb{N}$ and $a \in \mathbb{R}_+\backslash\{1\}$, evaluate:

$$\lim_{n\to\infty} n^k(a^{\frac{1}{n}}-1)\left(\sqrt{\frac{n-1}{n}}-\sqrt{\frac{n+1}{n+2}}\right)$$

Solution:

$$\lim_{n\to\infty} n^k(a^{\frac{1}{n}}-1)\left(\sqrt{\frac{n-1}{n}}-\sqrt{\frac{n+1}{n+2}}\right) = \lim_{n\to\infty}\frac{-n^k(a^{\frac{1}{n}}-1)}{n(n+2)}\cdot\lim_{n\to\infty}\frac{2}{\sqrt{\frac{n-1}{n}}+\sqrt{\frac{n+1}{n+2}}}$$
$$= \lim_{n\to\infty}\frac{-n^{k-1}}{n(n+2)}\cdot\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}$$
$$= \ln a\cdot\lim_{n\to\infty}\frac{-n^{k-2}}{n+2}$$
$$= \begin{cases} 0 & \text{if } k\in\{0,1,2\} \\ -\ln a & \text{if } k=3 \\ \infty & \text{if } k\geq 4 \text{ and } a\in(0,1) \\ -\infty & \text{if } k\geq 4 \text{ and } a>1 \end{cases}$$

23. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$$

Solution: Clearly

$$\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+k}} \leq \frac{1}{\sqrt{n^2+1}},\ (\forall)1\leq k\leq n$$

Thus summing for $k=\overline{1,n}$, we get:

$$\frac{n}{\sqrt{n^2+n}} \leq \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}} \leq \frac{n}{\sqrt{n^2+1}}$$

Because $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$ and $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} =$ 1, using the squeeze theorem it follows that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = 1$$

24. If $a > 0,\ p \geq 2$, evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt[p]{n^p+ka}}$$

Solution: Obviously

$$\frac{1}{\sqrt[p]{n^p+na}} \leq \frac{1}{\sqrt[p]{n^p+ka}} \leq \frac{1}{\sqrt[p]{n^p+a}},\ (\forall) 1 \leq k \leq n$$

Thus summing for $k = \overline{1,n}$, we get:

$$\frac{n}{\sqrt[p]{n^p+na}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \frac{n}{\sqrt[p]{n^p+a}}$$

Because $\lim_{n\to\infty} \frac{n}{\sqrt[p]{n^p+a}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\dfrac{a}{n^p}}} = 1$ and $\lim_{n\to\infty} \frac{n}{\sqrt[p]{n^p+na}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\dfrac{a}{n^{p-1}}}} =$ 1, using the squeeze theorem it follows that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt[p]{n^p+ka}} = 1$$

25. Evaluate:

$$\lim_{n\to\infty} \frac{n!}{(1+1^2)(1+2^2)\cdot\ldots\cdot(1+n^2)}$$

Solution: We have

$$\begin{aligned} 0 &\leq \frac{n!}{(1+1^2)(1+2^2)\cdot\ldots\cdot(1+n^2)} \\ &< \frac{n!}{1^2\cdot 2^2\cdot\ldots\cdot n^2} \\ &= \frac{n!}{(1\cdot 2\cdot\ldots\cdot n)\cdot(1\cdot 2\cdot\ldots\cdot n)} \\ &= \frac{n!}{(n!)^2} \\ &= \frac{1}{n!} \end{aligned}$$

Thus using squeeze theorem it follows that:

$$\lim_{n\to\infty} \frac{n!}{(1+1^2)(1+2^2)\cdot\ldots\cdot(1+n^2)} = 0$$

26. Evaluate:

$$\lim_{n\to\infty} \left(\frac{2n^2-3}{2n^2-n+1}\right)^{\dfrac{n^2-1}{n}}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{2n^2-3}{2n^2-n+1}\right)^{\frac{n^2-1}{n}} &= \lim_{n\to\infty} \left(1+\frac{n-4}{2n^2-n+1}\right)^{\frac{n^2-1}{n}} \\
&= \lim_{n\to\infty} \left[\left(1+\frac{n-4}{2n^2-n+1}\right)^{\frac{2n^2-n+1}{n-4}}\right]^{\frac{(n-4)(n^2-1)}{2n^3-2n^2+n}} \\
&= e^{\lim\limits_{n\to\infty} \frac{n^3-4n^2-n+4}{2n^3-2n^2+n}} \\
&= e^{\frac{1}{2}} \\
&= \sqrt{e}
\end{aligned}$$

27. Evaluate:

$$\lim_{x\to 0} \frac{\sqrt{1+\sin^2 x}-\cos x}{1-\sqrt{1+\tan^2 x}}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\sin^2 x}-\cos x}{1-\sqrt{1+\tan^2 x}} &= \lim_{x\to 0} \frac{(1+\sin^2 x-\cos^2 x)(1+\sqrt{1+\tan^2 x})}{(1-1-\tan^2 x)(\sqrt{1+\sin^2 x}+\cos x)} \\
&= \lim_{x\to 0} \frac{2\sin^2 x(1+\sqrt{1+\tan^2 x})}{-\tan^2 x(\sqrt{1+\sin^2 x}+\cos x)} \\
&= \lim_{x\to 0} \frac{-2\cos^2 x(1+\sqrt{1+\tan^2 x})}{\sqrt{1+\sin^2 x}+\cos x} \\
&= -2
\end{aligned}$$

28. Evaluate:

$$\lim_{x\to\infty} \left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right)^x$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right)^x &= \lim_{x\to\infty}\left(1+\frac{2\sqrt{x}}{x-\sqrt{x}}\right)^x \\
&= \lim_{x\to\infty}\left[\left(1+\frac{2\sqrt{x}}{x-\sqrt{x}}\right)^{\dfrac{x-\sqrt{x}}{2\sqrt{x}}}\right]^{\dfrac{2x\sqrt{x}}{x-\sqrt{x}}} \\
&= e^{\lim\limits_{x\to\infty}\dfrac{2x\sqrt{x}}{x-\sqrt{x}}} \\
&= e^{\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}}{1-\frac{1}{\sqrt{x}}}} \\
&= e^{\infty} \\
&= \infty
\end{aligned}$$

29. Evaluate:

$$\lim_{\substack{x\to 0\\ x>0}}(\cos x)^{\dfrac{1}{\sin x}}$$

Solution:

$$\begin{aligned}
\lim_{\substack{x\to 0\\ x>0}}(\cos x)^{\dfrac{1}{\sin x}} &= \lim_{\substack{x\to 0\\ x>0}}\left[(1+(\cos x-1))^{\dfrac{1}{\cos x-1}}\right]^{\dfrac{\cos x-1}{\sin x}} \\
&= e^{\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}} \\
&= e^{\lim\limits_{\substack{x\to 0\\ x>0}}-\tan\dfrac{x}{2}} \\
&= e^0 \\
&= 1
\end{aligned}$$

30. Evaluate:

$$\lim_{x\to 0}\left(e^x+\sin x\right)^{\frac{1}{x}}$$

Solution:

$$\begin{aligned}\lim_{x\to 0}(e^x+\sin x)^{\frac{1}{x}} &= \lim_{x\to 0}\left[e^x\left(1+\frac{\sin x}{e^x}\right)\right]^{\frac{1}{x}}\\ &= \lim_{x\to 0}(e^x)^{\frac{1}{x}}\cdot\lim_{x\to 0}\left[\left(1+\frac{\sin x}{e^x}\right)^{\dfrac{e^x}{\sin x}}\right]^{\dfrac{\sin x}{xe^x}}\\ &= e\cdot e^{\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot\dfrac{1}{e^x}}\\ &= e^2\end{aligned}$$

31. If $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n &= \lim_{n\to\infty}\left[\left(1+\frac{\sqrt[n]{b}-1}{a}\right)^{\dfrac{a}{\sqrt[n]{b}-1}}\right]^{\dfrac{n(\sqrt[n]{b}-1)}{a}}\\ &= e^{\dfrac{1}{a}\lim\limits_{n\to\infty}\dfrac{b^{\frac{1}{n}}-1}{\frac{1}{n}}}\\ &= e^{\dfrac{\ln b}{a}}\\ &= b^{\frac{1}{a}}\end{aligned}$$

32. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ defined by:

$$a_n = \begin{cases} 1 & \text{if } n \leq k,\ k \in \mathbb{N}^* \\ \dfrac{(n+1)^k - n^k}{\binom{n}{k-1}} & \text{if } n > k \end{cases}$$

i)Evaluate $\lim\limits_{n\to\infty} a_n$.

ii)If $b_n = 1 + \sum\limits_{k=1}^{n} k \cdot \lim\limits_{n\to\infty} a_n$, evaluate:

$$\lim_{n\to\infty}\left(\frac{b_n^2}{b_{n-1}b_{n+1}}\right)^n$$

Solution: i) We have

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(n+1)^k - n^k}{\begin{pmatrix} n \\ k-1 \end{pmatrix}} \\ &= \lim_{n\to\infty} \frac{(k-1)! \cdot k \cdot n^{k-1} + \ldots + (k-1)!}{(n-k+2)(n-k+3)\cdot \ldots \cdot n} \\ &= \frac{k! \cdot n^{k-1} + \ldots}{n^{k-1} + \ldots} \\ &= k!\end{aligned}$$

ii) Then:

$$b_n = 1 + \sum_{k=1}^{n} k \cdot k! = 1 + \sum_{k=1}^{n} (k+1)! - \sum_{k=1}^{n} k! = (n+1)!$$

so

$$\lim_{n\to\infty} \left(\frac{b_n^2}{b_{n-1} b_{n+1}} \right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$$

33. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_{n+2} = \dfrac{x_{n+1} + x_n}{2}$, $(\forall) n \in \mathbb{N}^*$. If $x_1 \leq x_2$,

i)Prove that the sequence $(x_{2n+1})_{n\geq 0}$ is increasing, while the sequence $(x_{2n})_{n\geq 0}$ is decreasing;

ii)Prove that:

$$|x_{n+2} - x_{n+1}| = \frac{|x_2 - x_1|}{2^n},\ (\forall) n \in \mathbb{N}^*$$

iii)Prove that:

$$2x_{n+2} + x_{n+1} = 2x_2 + x_1,\ (\forall) n \in \mathbb{N}^*$$

iv)Prove that $(x_n)_{n\geq 1}$ is convergent and that it's limit is $\dfrac{x_1 + 2x_2}{3}$.

Solution: i)Using induction we can show that $x_{2n-1} \leq x_{2n}$. Then the sequence $(x_{2n+1})_{n\geq 0}$ will be increasing, because

$$x_{2n+1} = \frac{x_{2n} + x_{2n-1}}{2} \geq \frac{x_{2n-1} + x_{2n-1}}{2} = x_{2n-1}$$

Similarly, we can show that $(x_{2n})_{n\geq 1}$ is decreasing.

ii) For $n = 1$, we get $|x_3 - x_2| = \dfrac{|x_2 - x_1|}{2}$, so assuming it's true for some k, we have:

$$|x_{k+3} - x_{k+2}| = \left| \frac{x_{k+2} + x_{k+1}}{2} - x_{k+2} \right| = \frac{|x_{k+2} - x_{k+1}|}{2} = \frac{|x_2 - x_1|}{2^{n+1}}$$

Thus, by induction the equality is proven.

iii) Observe that:

$$2x_{n+2} + x_{n+1} = 2 \cdot \frac{x_{n+1} + x_n}{2} + x_{n+1} = 2x_{n+1} + x_n$$

and repeating the process, the demanded identity is showed.

iv) From i) it follows that the sequences $(x_{2n})_{n\geq 1}$ and $(x_{2n-1})_{n\geq 1}$ are convergent and have the same limit. Let $l = \lim\limits_{n\to\infty} x_n = l$. Then from iii), we get

$$3l = x_1 + 2x_2 \Rightarrow l = \frac{x_1 + 2x_2}{3}$$

34. Let $a_n, b_n \in \mathbb{Q}$ such that $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$, $(\forall)n \in \mathbb{N}^*$. Evaluate $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$.

Solution: Because $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$, it follows that $(1-\sqrt{2})^n = a_n - b_n\sqrt{2}$. Solving this system we find:

$$a_n = \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]$$

and

$$b_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

and therefore $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$.

35. If $a > 0$, evaluate:

$$\lim_{x\to 0} \frac{(a+x)^x - 1}{x}$$

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{(a+x)^x - 1}{x} &= \lim_{x\to 0} \frac{e^{x\ln(a+x)} - 1}{x} \\ &= \lim_{x\to 0} \frac{e^{x\ln(a+x)} - 1}{x\ln(a+x)} \cdot \lim_{x\to 0} \ln(a+x) \\ &= \ln a \end{aligned}$$

36. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ such that $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{a_n^2 - a_n + 1}{a_n}$. Prove that $(a_n)_{n\geq 1}$ is convergent and find it's limit.

Solution: By $AM-GM$ we have $a_{n+1}=a_n+\frac{1}{a_n}-1\geq 1,\ (\forall)n\geq 2$, so the sequence is lower bounded. Also $a_{n+1}-a_n=\frac{1}{a_n}-1\leq 0$, hence the sequence is decreasing. Therefore $(a_n)_{n\geq 1}$ is bounded by 1 and $a_1=\frac{3}{2}$. Then, because $(a_n)_{n\geq 1}$ is convergent, denote $\lim\limits_{n\to\infty} a_n=l$, to obtain $l=\frac{l^2-l+1}{l}\Rightarrow l=1$

37. Consider sequence $(x_n)_{n\geq 1}$ of real numbers such that $x_0\in(0,1)$ and $x_{n+1}=x_n-x_n^2+x_n^3-x_n^4,\ (\forall)n\geq 0$. Prove that this sequence is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

Solution: It's easy to see that the recurrence formula can be written as: $x_{n+1}=x_n(1-x_n)(1+x_n^2),\ n\in\mathbb{N}$, then because $1-x_0>0$, it's easy to show by induction that $x_n\in(0,1)$. Now rewrite the recurrence formula as $x_{n+1}-x_n=-x_n^2(x_n^2-x_n+1)<0$. It follows that the sequence is strictly decreasing, thus convergent. Let $\lim\limits_{n\to\infty} x_n=l$. Then

$$l=l-l^2+l^3-l^4\Rightarrow l^2(l^2-l+1)=0\Rightarrow l=0$$

38. Let $a>0$ and $b\in(a,2a)$ and a sequence $x_0=b,\ x_{n+1}=a+\sqrt{x_n(2a-x_n)},\ (\forall)n\geq 0$. Study the convergence of the sequence $(x_n)_{n\geq 0}$.

Solution: Let's see a few terms: $x_1=a+\sqrt{2ab-b^2}$ and also

$$x_2=a+\sqrt{(a+\sqrt{2ab-b^2})(a-\sqrt{2ab-b^2})}=a+\sqrt{a^2-2ab+b^2}=a+|a-b|=b$$

Thus the sequence is periodic, with $x_{2k}=b$ and $x_{2k+1}=a+\sqrt{2ab-b^2},\ (\forall)k\in\mathbb{N}$. Then $\lim\limits_{k\to\infty} x_{2k}=b$ and $\lim\limits_{n\to\infty} x_{2k+1}=a+\sqrt{2ab-b^2}$. The sequence is convergent if and only if $b=a+\sqrt{2ab-b^2}$, which implies that $b=\left(1+\frac{1}{\sqrt{2}}\right)a$, which is also the limit in this case.

39. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n+1}\arctan\frac{1}{2k^2}$$

Solution: We can check easily that $\arctan\frac{1}{2k^2}=\arctan\frac{k}{k+1}-\arctan\frac{k-1}{k}$. Then:

$$\lim_{n\to\infty}\sum_{k=1}^{n+1}\arctan\frac{1}{2k^2}=\lim_{n\to\infty}\arctan\frac{n}{n+1}=\frac{\pi}{4}$$

40. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{4k^4+1}$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{4k^4+1} &= \lim_{n\to\infty}\left(\frac{1}{2}\sum_{k=1}^{n}\frac{1}{2k^2-2k+1}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{2k^2+2k+1}\right)\\ &= \frac{1}{4}\lim_{n\to\infty}\left(1-\frac{1}{2n^2+2n+1}\right)\\ &= \frac{1}{4}\end{aligned}$$

41. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+3+3^2+\ldots+3^k}{5^{k+2}}$$

Solution: In the numerator we have a geometrical progression, so:

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+3+3^2+\ldots+3^k}{5^{k+2}} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{3^{k+1}-1}{2\cdot 5^{k+2}}\\ &= \frac{1}{10}\lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{3^k}{5^k}-\frac{1}{5^k}\right)\\ &= \frac{1}{10}\left(\frac{9}{10}-\frac{1}{20}\right)\\ &= \frac{17}{200}\end{aligned}$$

42. Evaluate:

$$\lim_{n\to\infty}\left(n+1-\sum_{i=2}^{n}\sum_{k=2}^{i}\frac{k-1}{k!}\right)$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\left(n+1-\sum_{i=2}^{n}\sum_{k=2}^{i}\frac{k-1}{k!}\right) &= \lim_{n\to\infty}\left(n+1-\sum_{i=2}^{n}\sum_{k=2}^{i}\left(\frac{1}{(k-1)!}-\frac{1}{k!}\right)\right)\\ &= \lim_{n\to\infty}\left(n+1-\sum_{i=2}^{n}\left(1-\frac{1}{i!}\right)\right)\\ &= \lim_{n\to\infty}\left(1+\sum_{i=1}^{n}\frac{1}{i!}\right)^{n}\\ &= e\end{aligned}$$

43. Evaluate:

$$\lim_{n\to\infty} \frac{1^1+2^2+3^3+\ldots+n^n}{n^n}$$

Solution: Using Cesaro-Stolz theorem we have:

$$\begin{aligned}\lim_{n\to\infty} \frac{1^1+2^2+3^3+\ldots+n^n}{n^n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1}-n^n} \\ &= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}-\frac{1}{n}} \\ &= \frac{e}{e-0} \\ &= 1\end{aligned}$$

44. Consider the sequence $(a_n)_{n\geq 1}$ such that $a_0 = 2$ and $a_{n-1} - a_n = \frac{n}{(n+1)!}$. Evaluate $\lim_{n\to\infty} ((n+1)!\ln a_n)$.

Solution: Observe that

$$a_k - a_{k-1} = \frac{-k}{(k+1)!} = \frac{1}{(k+1)!} - \frac{1}{k!},\ (\forall) 1 \leq k \leq n$$

Letting $k = 1, 2, 3 \cdots n$ and summing, we get $a_n - a_0 = \frac{1}{(n+1)!} - 1$. Since $a_0 = 2$ we get $a_n = 1 + \frac{1}{(n+1)!}$. Using the result $\lim_{f(x)\to 0} \frac{\ln(1+f(x))}{f(x)} = 1$, we conclude that

$$\lim_{n\to\infty} (n+1)!\ln a_n = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{(n+1)!}\right)}{\frac{1}{(n+1)!}} = 1$$

45. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ with $x_1 = a > 0$ and $x_{n+1} = \frac{x_1+2x_2+3x_3+\ldots+nx_n}{n}$, $n \in \mathbb{N}^*$. Evaluate it's limit.

Solution: The sequence is strictly increasing because:

$$x_{n+1}-x_n = \frac{x_1+2x_2+3x_3+\ldots+nx_n}{n} - x_n = \frac{x_1+2x_2+3x_3+\ldots+(n-1)x_{n-1}}{n} > 0$$

Then

$$x_{n+1} > \frac{a+2a+\ldots+na}{n} = \frac{(n+1)a}{2}$$

It follows that $\lim_{n\to\infty} x_n = \infty$.

46. Using $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}$, evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)^2}$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)^2} &= \lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{k^2}-\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k)^2}\\ &= \lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{k^2}-\frac{1}{4}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}\\ &= \frac{\pi}{6}-\frac{\pi}{24}\\ &= \frac{\pi}{8}\end{aligned}$$

47. Consider the sequence $(x_n)_{n\geq 1}$ defined by $x_1=a,\ x_2=b,\ a<b$ and $x_n=\frac{x_{n-1}+\lambda x_{n-2}}{1+\lambda},\ n\geq 3,\ \lambda>0$. Prove that this sequence is convergent and find it's limit.

Solution: The sequence isn't monotonic because $x_3=\frac{b+\lambda a}{1+\lambda}\in[a,b]$. We can prove by induction that $x_n\in[a,b]$. The sequences $(x_{2n})_{n\geq 1}$ and $(x_{2n-1})_{n\geq 1}$ are monotonically increasing. Also, we can show by induction, that:

$$x_{2k}-x_{2k-1}=\left(\frac{\lambda}{1+\lambda}\right)^{2k}(b-a)$$

It follows that the sequences $(x_{2n})_{n\geq 1}$ and $(x_{2n-1})_{n\geq 1}$ have the same limit, so $(x_n)_{n\geq 1}$ is convergent. The recurrence formulas can be written as

$$x_k-x_{k-1}=\lambda(x_{k-2}-x_k),\ (\forall)k\geq 3$$

Summing for $k=3,4,5,\ldots,n$, we have:

$$x_n-b=\lambda(a+b-x_{n-1}-x_n)\Leftrightarrow(1+\lambda)x_n+\lambda x_{n-1}=(1+\lambda)b+\lambda a$$

By passing to limit, it follows that:

$$\lim_{n\to\infty}x_n=\frac{b+\lambda(a+b)}{1+2\lambda}$$

48. Evaluate:

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$$

Solution: Using the consequence of Cesaro-Stolz lemma, we have:

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} &= \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}\\ &= \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\\ &= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n\cdot(n+1)}\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\\ &= e\end{aligned}$$

49. Consider the sequence $(x_n)_{n\geq 1}$ defined by $x_1 = 1$ and $x_n = \dfrac{1}{1+x_{n-1}}$, $n \geq 2$. Prove that this sequence is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

Solution: We can show easily by induction that $x_n \in (0,1)$ and that the sequence $(x_{2n})_{n\geq 1}$ is increasing, while the sequence $(x_{2n-1})_{n\geq 1}$ is decreasing. Observe that:

$$x_{2n+2} = \frac{1}{1+x_{2n+1}} = \frac{1}{1+\frac{1}{1+x_{2n}}} = \frac{1+x_{2n}}{2+x_{2n}}$$

The sequence $(x_{2n})_{n\geq 1}$ is convergent, so it has the limit $\dfrac{\sqrt{5}-1}{2}$. Similarly $\lim\limits_{n\to\infty} x_{2n-1} = \dfrac{\sqrt{5}-1}{2}$. Therefore $(x_n)_{n\geq 1}$ is convergent and has the limit equal to $\dfrac{\sqrt{5}-1}{2}$.

50. If $a, b \in \mathbb{R}^*$, evaluate:

$$\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)} &= \lim_{x\to 0}\frac{(\cos ax-1)\cdot\ln(1+\cos ax-1)^{\frac{1}{\cos ax-1}}}{(\cos bx-1)\cdot\ln(1+\cos bx-1)^{\frac{1}{\cos bx-1}}}\\ &= \lim_{x\to 0}\frac{-2\sin^2\frac{ax}{2}}{-2\sin^2\frac{bx}{2}}\\ &= \frac{a^2}{b^2}\end{aligned}$$

51. Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}\{x\} & \text{if } x\in\mathbb{Q}\\ x & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}$. Find all $\alpha\in\mathbb{R}$ for which $\lim\limits_{x\to\alpha} f(x)$ exists.

Solution: Let $f=g-h$, where $g:\mathbb{R}\to\mathbb{R},\ g(x)=x,\ (\forall)x\in\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R},\ h(x)=\begin{cases}\lfloor x\rfloor & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}$. If $\alpha\in\mathbb{R}\backslash[0,1)$, we can find two sequences $x_n\in\mathbb{Q}$ and $y_n\in\mathbb{R}\backslash\mathbb{Q}$ going to α, such that the sequences $(f(x_n))$ and $(f(y_n))$ have different limits. If $\alpha\in[0,1)$, $h(x)=0$ and $f(x)=x$, thus $(\forall)\alpha\in[0,1)$, we have $\lim\limits_{x\to\alpha} f(x)=\alpha$.

52. Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}\lfloor x\rfloor & \text{if } x\in\mathbb{Q}\\ x & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}$. Find all $\alpha\in\mathbb{R}$ for which $\lim\limits_{x\to\alpha} f(x)$ exists.

Solution: Divide the problem in two cases:

Case I: $\alpha=k\in\mathbb{Z}$. Consider a sequence $(x_n),\ x_n\in(k-1,k)\cap\mathbb{Q}$ and $(y_n),\ y_n\in(k-1,k)\cap(\mathbb{R}\backslash\mathbb{Q})$, both tending to k. Then:

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}\lfloor x_n\rfloor=\lim_{n\to\infty}(k-1)=k-1$$

and $\lim\limits_{n\to\infty} f(y_n)=\lim\limits_{n\to\infty} y_n=k$. Therefore $\lim\limits_{x\to\alpha} f(x)$ doesn't exist.

Case II: $\alpha\in\mathbb{R}\backslash\mathbb{Z}$. Let $\lfloor\alpha\rfloor=k$. Consider a sequence $(x_n),\ x_n\in(k,k+1)\cap\mathbb{Q}$ and $(y_n),\ y_n\in(k,k+1)\cap(\mathbb{R}\backslash\mathbb{Q})$, which tend both to α. Then:

$$\lim_{n\to\alpha} f(x_n)=\lim_{n\to\alpha}\lfloor x_n\rfloor=\lim_{n\to\alpha} k=k$$

and $\lim\limits_{n\to\alpha} f(y_n)=\lim\limits_{n\to\alpha} y_n=\alpha$. Again, in this case, $\lim\limits_{x\to\alpha} f(x)$ doesn't exist.

53. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that $x_1>0$ and $3x_n=2x_{n-1}+\dfrac{a}{x_{n-1}^2}$, where a is a real positive number. Prove that x_n is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

Solution: By AM-GM

$$x_{n+1}=\frac{x_n+x_n+\frac{a}{x_n^2}}{3}\ \geq\ \sqrt[3]{x_n\cdot x_n\cdot\frac{a}{x_n^2}}=\sqrt[3]{a}\Rightarrow\ x_n\geq\sqrt[3]{a}$$

Also

$$3(x_{n+1}-x_n)=\frac{a}{x_n^2}-x_n=\frac{a-x_n^3}{x_n^3}\leq 0\Rightarrow x_{n+1}-x_n\leq 0,\forall\ n\in\mathbb{N}\ ,\ n\geq 2$$

Therefore, the sequence $(x_n)_{n\geq 1}$ is decreasing and lower bounded, so it's convergent. By passing to limit in the recurrence formula we obtain $\lim\limits_{n\to\infty} x_n = \sqrt[3]{a}$.

54. Consider a sequence of real numbers $(a_n)_{n\geq 1}$ such that $a_1 = 12$ and $a_{n+1} = a_n\left(1+\dfrac{3}{n+1}\right)$. Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{a_k}$$

Solution: Rewrite the recurrence formula as

$$a_{n+1} = a_n \cdot \frac{n+4}{n+1}$$

Writing it for $n = 1, 2, \ldots, n-1$ and multiplying the obtained equalities, we find that:

$$a_n = \frac{(n+1)(n+2)(n+3)}{2},\ (\forall) n \in \mathbb{N}^*$$

Then:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{a_k} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{(k+1)(k+2)(k+3)}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{2}{k+2}+\frac{1}{k+3}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{6}-\frac{1}{n+2}+\frac{1}{n+3}\right)\\
&= \frac{1}{6}
\end{aligned}$$

55. Evaluate:

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+1}}\right)^n$$

Solution:

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+1}}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{n-\sqrt{n^2+1}}{\sqrt{n^2+1}}\right)^{\frac{\sqrt{n^2+1}}{n-\sqrt{n^2+1}}}\right]^{\frac{n\left(n-\sqrt{n^2+1}\right)}{\sqrt{n^2+1}}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{n\left(n-\sqrt{n^2+1}\right)}{\sqrt{n^2+1}}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{-n}{\sqrt{n^2+1}\cdot\left(\sqrt{n^2+1}+n\right)}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{-n}{n^2+1+n\sqrt{n^2+1}}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{-1}{n+\frac{1}{n}+\sqrt{n^2+1}}}$$

$$= e^0$$

$$= 1$$

56. If $a \in \mathbb{R}$, evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\lfloor k^2 a\rfloor}{n^3}$$

Solution: We have $x-1<\lfloor x\rfloor\le x,\ (\forall)x\in\mathbb{R}$. Choosing $x=k^2a$, letting k to take values from 1 to n and summing we have:

$$\sum_{k=1}^{n}(k^2a-1)<\sum_{k=1}^{n}\lfloor k^2a\rfloor\le\sum_{k=1}^{n}k^2a \Leftrightarrow \frac{\sum\limits_{k=1}^{n}(k^2a-1)}{n^3}<\frac{\sum\limits_{k=1}^{n}\lfloor k^2a\rfloor}{n^3}\le\frac{\sum\limits_{k=1}^{n}k^2a}{n^3}$$

Now observe that:

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}(k^2a-1)}{n^3}=\lim_{n\to\infty}\frac{a\cdot\frac{n(n+1)(2n+1)}{6}-n}{n^3}=\frac{a}{3}$$

and

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}k^2a}{n^3}=\lim_{n\to\infty}\frac{an(n+1)(2n+1)}{6n^3}=\frac{a}{3}$$

So using the Squeeze Theorem it follows that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lfloor k^2 a \rfloor}{n^3} = \frac{a}{3}$$

57. Evaluate:

$$\lim_{n\to\infty} 2^n \left(\sum_{k=1}^{n} \frac{1}{k(k+2)} - \frac{1}{4} \right)^n$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} 2^n \left(\sum_{k=1}^{n} \frac{1}{k(k+2)} - \frac{1}{4} \right)^n &= \lim_{n\to\infty} 2^n \left(\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{4} \right)^n \\
&= \lim_{n\to\infty} 2^n \left(\frac{3}{4} - \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n+2} \right) - \frac{1}{4} \right)^n \\
&= \lim_{n\to\infty} \left(1 - \frac{2n+3}{(n+1)(n+2)} \right)^n \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{2n+3}{(n+1)(n+2)} \right)^{\frac{-(n+1)(n+2)}{2n+3}} \right]^{\frac{-n(2n+3)}{(n+1)(n+2)}} \\
&= e^{\lim\limits_{n\to\infty} \frac{-2n^2-3n}{n^2+3n+2}} \\
&= e^{-2}
\end{aligned}$$

58. Consider the sequence $(a_n)_{n\geq 1}$, such that $a_n > 0$, $(\forall) n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} n(a_{n+1} - a_n) = 1$. Evaluate $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$.

Solution: Start with the ε criterion

$$\lim_{n\to\infty} n(a_{n+1} - a_n) = 1 \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists) n_\varepsilon \in \mathbb{N},\ (\forall) n \geq n_\varepsilon \Rightarrow |n(a_{n+1} - a_n) - 1| < \varepsilon$$

Let $\varepsilon \in (0, 1)$. Then for $n \geq n_\varepsilon$, we have:

$$-\varepsilon < n(a_{n+1} - a_n) - 1 < \varepsilon \Rightarrow \frac{1-\varepsilon}{n} < a_{n+1} - a_n < \frac{1+\varepsilon}{n}$$

Summing for $n = n_\varepsilon, n_\varepsilon + 1, \ldots, n$, we get:

$$(1-\varepsilon) \left(\frac{1}{n_\varepsilon} + \frac{1}{n_\varepsilon + 1} + \ldots + \frac{1}{n} \right) < a_{n+1} - a_{n_\varepsilon} < (1-\varepsilon) \left(\frac{1}{n_\varepsilon} + \frac{1}{n_\varepsilon + 1} + \ldots + \frac{1}{n} \right)$$

By passing to limit, it follows that $\lim_{n\to\infty} a_n = \infty$. To evaluate $\lim_{n\to\infty} \sqrt[n]{a_n}$, recall that in the above conditions we have:

$$\frac{1-\varepsilon}{n} < a_{n+1} - a_n < \frac{1+\varepsilon}{n} \Rightarrow \frac{1-\varepsilon}{na_n} < \frac{a_{n+1}}{a_n} - 1 < \frac{1+\varepsilon}{na_n}$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, and the root test implies that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$

59. Evaluate:

$$\lim_{n\to\infty} \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \ldots + n\sqrt{n}}{n^2\sqrt{n}}$$

Solution: Using Cesaro-Stolz lemma, we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \ldots + n\sqrt{n}}{n^2\sqrt{n}} &= \lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}}{(n+1)^2\sqrt{n+1} - n^2\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{(n+1)^3}}{\sqrt{(n+1)^5} - \sqrt{n^5}} \\
&= \lim_{n\to\infty} \frac{\sqrt{(n+1)^3}\left(\sqrt{(n+1)^5} + \sqrt{n^5}\right)}{(n+1)^5 - n^5} \\
&= \lim_{n\to\infty} \frac{n^4 + 4n^3 + 6n^2 + 4n + 1 + \sqrt{n^8 + 3n^7 + 3n^6 + n^5}}{5n^4 + 10n^3 + 10n^2 + 5n + 1} \\
&= \lim_{n\to\infty} \frac{1 + \frac{4}{n} + \frac{6}{n^2} + \frac{4}{n^3} + \frac{1}{n^4} + \sqrt{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}}{5 + \frac{10}{n} + \frac{10}{n^2} + \frac{5}{n^3} + \frac{1}{n^4}} \\
&= \frac{2}{5}
\end{aligned}$$

60. Evaluate:

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\frac{1}{2x-\pi}}$$

Solution:

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{2x-\pi}} &= \lim_{x\to\frac{\pi}{2}}\left[(1+\sin x-1)^{\frac{1}{\sin x-1}}\right]^{\frac{\sin x-1}{2x-\pi}}\\ &= e^{\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-1}{2x-\pi}}\\ &= e^{\lim\limits_{y\to 0}\frac{\cos y-1}{2y}}\\ &= e^{\lim\limits_{y\to 0}\frac{-\sin^2\frac{y}{2}}{y}}\\ &= e^{\lim\limits_{y\to 0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\cdot\left(-\frac{y}{4}\right)}\\ &= e^0\\ &= 1\end{aligned}$$

61. Evaluate:

$$\lim_{n\to\infty} n^2\ln\left(\cos\frac{1}{n}\right)$$

Solution: We'll use the well-known limit $\lim\limits_{x_n\to 0}\dfrac{\ln(1+x_n)}{x_n}=1$. We have:

$$\begin{aligned}\lim_{n\to\infty} n^2\ln\left(\cos\frac{1}{n}\right) &= \lim_{n\to\infty}\left[n^2\left(\cos\frac{1}{n}-1\right)\right]\cdot\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}-1\right)}{\cos\frac{1}{n}-1}\\ &= \lim_{n\to\infty} -2n^2\cdot\sin^2\frac{1}{2n}\\ &= \lim_{n\to\infty} -\frac{1}{2}\cdot\left(\frac{\sin\frac{1}{2n}}{\frac{1}{2n}}\right)^2\\ &= -\frac{1}{2}\end{aligned}$$

62. Given $a,b\in\mathbb{R}_+^*$, evaluate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$$

Solution: Using the limits $\lim\limits_{x_n\to\infty}(1+x_n)^{\frac{1}{x_n}}=e$ and $\lim\limits_{n\to\infty} n(\sqrt[n]{a}-1)=\ln a$, we have:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1+\sqrt[n]{b}-1}{2}\right)^n$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{\sqrt[n]{a}-1+\sqrt[n]{b}-1}{2}\right)^{\frac{2}{\sqrt[n]{a}-1+\sqrt[n]{b}-1}}\right]^{\frac{n(\sqrt[n]{a}-1)+n(\sqrt[n]{b}-1)}{2}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{n(\sqrt[n]{a}-1)+n(\sqrt[n]{b}-1)}{2}}$$

$$= e^{\frac{\ln a+\ln b}{2}}$$

$$= e^{\ln\sqrt{ab}}$$

$$= \sqrt{ab}$$

63. Let $\alpha > \beta > 0$ and the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

i)Prove that $(\exists)(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \mathbb{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^n = x_n A + y_n B,\ (\forall) n \geq 1$$

ii)Evaluate $\lim\limits_{n\to\infty} \frac{x_n}{y_n}$.

Solution: i) We proceed by induction. For $n = 1$, we have

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} = \alpha A + \beta B$$

Hence $x_1 = \alpha$ and $y_1 = \beta$. Let

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^k = x_k A + y_k B$$

Then

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^{k+1} = (\alpha A + \beta B)(x_k A + y_k B)$$

Using $B^2 = A$, we have:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^{k+1} = (\alpha x_k + \beta y_k)A + (\beta x_k + \alpha y_k)B$$

Thus $x_{k+1} = \alpha x_k + \beta y_k$ and $y_{k+1} = \beta x_k + \alpha y_k$.

ii) An easy induction shows that $x_n, y_n > 0,\ (\forall)n \in \mathbb{N}^*$. Let $X \in \mathcal{M}_2(\mathbb{R})$ such that $X^n = \begin{pmatrix} x_n & y_n \\ y_n & x_n \end{pmatrix}$. Because $\det(X^n) = (\det X)^n$, it follows that $(\alpha^2 - \beta^2)^n = x_n^2 - y_n^2$, and because $\alpha > \beta$, we have $x_n > y_n,\ (\forall)n \in \mathbb{N}^*$. Let $z_n = \dfrac{x_n}{y_n}$. Then:

$$z_{n+1} = \frac{x_{n+1}}{y_{n+1}} = \frac{\alpha x_n + \beta y_n}{\beta x_n + \alpha y_n} = \frac{\alpha z_n + \beta}{\beta z_n + \alpha}$$

It's easy to see that the sequence is bounded by 1 and $\dfrac{\alpha}{\beta}$. Also the sequence is strictly decreasing, because

$$z_{n+1} - z_n = \frac{\alpha z_n + \beta}{\beta z_n + \alpha} - z_n = \frac{\beta(1 - z_n^2)}{\beta z_n + \alpha} < 0$$

Therefore the sequence is convergent. Let $\lim\limits_{n\to\infty} z_n = l$, then

$$l = \frac{\alpha l + \beta}{\beta l + \alpha} \Rightarrow l^2 = 1$$

l can't be -1, because $z_n \in \left(1, \dfrac{\alpha}{\beta}\right)$, hence $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = 1$.

64. If $a \in \mathbb{R}$ such that $|a| < 1$ and $p \in \mathbb{N}^*$ is given, evaluate:

$$\lim_{n\to\infty} n^p \cdot a^n$$

Solution: If $a = 0$, we get $n^p \cdot a^n = 0,\ (\forall)n \in \mathbb{N}$. If $a \neq 0$, since $|a| < 1$, there is a $\alpha > 0$ such that $|a| = \dfrac{1}{1+\alpha}$. Let now $n > p$, then from binomial expansion we get:

$$(1+\alpha)^n > C_n^{p+1} \cdot \alpha^{p+1} \Leftrightarrow \frac{1}{(1+\alpha)^n} < \frac{(p+1)!}{n(n-1)(n-2)\cdot \ldots \cdot (n-p)\cdot \alpha^{p+1}}$$

Then:

$$\begin{aligned} 0 &< |n^p \cdot a^n| \\ &= n^p \cdot |a|^n \\ &< \frac{n^p \cdot (p+1)!}{n(n-1)(n-2)\cdot \ldots \cdot (n-p)\cdot \alpha^{p+1}} \\ &= \frac{n^{p-1} \cdot (p+1)!}{(n-1)(n-2)\cdot \ldots \cdot (n-p)\cdot \alpha^{p+1}} \end{aligned}$$

Keeping in mind that

$$\lim_{n\to\infty} \frac{n^{p-1}\cdot (p+1)!}{(n-1)(n-2)\cdot\ldots\cdot(n-p)\cdot \alpha^{p+1}} = 0$$

and using the Squeeze Theorem, it follows that

$$\lim_{n\to\infty} n^p \cdot a^n = 0$$

65. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$$

Solution: Using Cesaro-Stolz lemma we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}} &= \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} \\
&= \lim_{n\to\infty} \frac{n^p + \binom{p}{1} n^{p-1} + \ldots}{\binom{p+1}{1} n^p + \binom{p+1}{2} n^{p-1} + \ldots} \\
&= \frac{1}{\binom{p+1}{1}} \\
&= \frac{1}{p+1}
\end{aligned}$$

66. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x\to 1\\ x<1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}}$$

First solution: Recall the identity:

$$\cos nt + i \sin nt = \binom{n}{0} \cos^n t + i \binom{n}{1} \cos^{n-1} t \cdot \sin t + \ldots + i^n \binom{n}{n} \sin^n t$$

For $t = \arccos x$, we have:

$$\sin(n \arccos x) = \binom{n}{1} x^{n-1} \cdot \sqrt{1-x^2} - \binom{n}{3} x^{n-3} (\sqrt{1-x^2})^3 + \binom{n}{5} x^{n-5} (\sqrt{1-x^2})^5 - \ldots$$

Then:

$$\lim_{\substack{x\to 1\\ x<1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}} = \lim_{\substack{x\to 1\\ x<1}} \left(\binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} (1-x^2) + \binom{n}{5} x^{n-5} (1-x^2)^2 - \ldots \right) = n$$

Second solution:

$$\begin{aligned}\lim_{\substack{x \to 1 \\ x<1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}} &= \lim_{\substack{x \to 1 \\ x<1}} \frac{\sin(n \arccos x)}{n \arccos x} \cdot \lim_{\substack{x \to 1 \\ x<1}} \frac{n \arccos x}{\sqrt{1-x^2}} \\ &= \lim_{\substack{x \to 1 \\ x<1}} \frac{n \arccos x}{\sqrt{1-x^2}} \\ &= \lim_{\substack{y \to 0 \\ y>0}} \frac{ny}{\sqrt{1-\cos^y}} \\ &= \lim_{\substack{y \to 0 \\ y>0}} \frac{ny}{\sin y} \\ &= n\end{aligned}$$

67. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \to 1 \\ x<1}} \frac{1-\cos(n \arccos x)}{1-x^2}$$

Solution:

$$\begin{aligned}\lim_{\substack{x \to 1 \\ x<1}} \frac{1-\cos(n \arccos x)}{1-x^2} &= \lim_{\substack{x \to 1 \\ x<1}} \frac{2\sin^2\left(\frac{n \arccos x}{2}\right)}{1-x^2} \\ &= \lim_{\substack{x \to 1 \\ x<1}} \frac{2\sin^2\left(\frac{n \arccos x}{2}\right)}{\left(\frac{n \arccos x}{2}\right)^2} \cdot \lim_{\substack{x \to 1 \\ x<1}} \frac{n^2 \arccos^2 x}{4(1-x^2)} \\ &= \lim_{\substack{x \to 1 \\ x<1}} \frac{n^2 \arccos^2 x}{4(1-x^2)} \\ &= \lim_{\substack{y \to 0 \\ y>0}} \frac{n^2 y^2}{2\sin^2 y} \\ &= \frac{n^2}{2}\end{aligned}$$

68. Study the convergence of the sequence:

$$x_{n+1} = \frac{x_n + a}{x_n + 1}, \ n \geq 1, \ x_1 \geq 0, \ a > 0$$

Solution: Consider a sequence $(y_n)_{n\geq 1}$ such that $x_n = \frac{y_{n+1}}{y_n} - 1$. Thus, our recurrence formula reduces to : $y_{n+2} - 2y_{n+1} + (1-a)y_n = 0$, whence $y_n = \alpha \cdot (1+\sqrt{a})^n + \beta \cdot (1-\sqrt{a})^n$. Finally:

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{\alpha\cdot(1+\sqrt{a})^{n+1}+\beta\cdot(1-\sqrt{a})^{n+1}}{\alpha\cdot(1+\sqrt{a})^{n}+\beta\cdot(1-\sqrt{a})^{n}} - 1 \\ &= \lim_{n\to\infty} \frac{\alpha\cdot(1+\sqrt{a})+\beta\cdot\left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^n\cdot(1-\sqrt{a})}{\alpha+\beta\cdot\left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^n} - 1 \\ &= \frac{\alpha\cdot(1+\sqrt{a})}{\alpha} - 1 \\ &= \sqrt{a}\end{aligned}$$

69. Consider two sequences of real numbers $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ such that $x_0 = y_0 = 3,\ x_n = 2x_{n-1} + y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1},\ (\forall)n \geq 1$. Evaluate $\lim\limits_{n\to\infty} \frac{x_n}{y_n}$.

First solution: Summing the hypothesis equalities, we have

$$x_n + y_n = 4(x_{n-1} + y_{n-1}),\ n \geq 1$$

Then $x_n + y_n = 4^(x_0 + y_0) = 6 \cdot 4^n$. Substracting the hypothesis equalities, we get

$$y_n - x_n = 2y_{n-1},\ n \geq 1$$

Summing with the previous equality we have $2y_n = 2y_{n-1} + 6\cdot 4^n \Rightarrow y_n - y_{n-1} = 3\cdot 4^n$. Then

$$y_1 - y_0 = 3\cdot 4$$

$$y_2 - y_1 = 3\cdot 4^2$$

$$y_3 - y_2 = 3\cdot 4^3$$

$$\dots$$

$$y_n - y_{n-1} = 3\cdot 4^n$$

Summing, it follows that:

$$y_n = y_0 + 3(4+4^2+\ldots+4^n) = 3(1+4+4^2+\ldots+4^n) = 3\cdot\frac{4^{n+1}-1}{4-1} = 4^{n+1}-1$$

Then $x_n = 2\cdot 4^n + 1$, and therefore:

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{2\cdot 4^n+1}{4\cdot 4^n-1} = \frac{1}{2}$$

Second solution: Define $a_n = \dfrac{x_n}{y_n}$ so that $a_n = \dfrac{2a_{n-1}+1}{2a_{n-1}+3}$. Now let $a_n = \dfrac{b_{n+1}}{b_n} - \dfrac{3}{2}$ to obtain $2b_{n+1} - 5b_n + b_{n-1} = 0$. Then $b_n = \alpha \cdot 2^n + \beta \cdot 2^{-n}$ for some $\alpha, \beta \in \mathbb{R}$. We finally come to:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\alpha \cdot 2^{n+1} + \beta \cdot 2^{-n-1}}{\alpha \cdot 2^n + \beta \cdot 2^{-n}} - \frac{3}{2} \right) = 2 - \frac{3}{2} = \frac{1}{2}$$

70. Evaluate:

$$\lim_{x\to 0} \frac{\tan x - x}{x^2}$$

Solution: If $x \in \left(0, \dfrac{\pi}{2}\right)$, we have:

$$0 < \frac{\tan x - x}{x^2} < \frac{\tan x - \sin x}{x^2} = \frac{\tan x(1-\cos x)}{x^2} = \frac{2\tan x \cdot \sin^2 \frac{x}{2}}{x^2}$$

and because $\displaystyle\lim_{x\to 0} \frac{2\tan x \cdot \sin^2 \frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{\tan x}{2} \cdot \lim_{x\to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 0$, using the Squeeze Theorem it follows that:

$$\lim_{\substack{x \to 0 \\ x>0}} \frac{\tan x - x}{x^2} = 0$$

Also

$$\lim_{\substack{x \to 0 \\ x<0}} \frac{\tan x - x}{x^2} = \lim_{\substack{y \to 0 \\ y>0}} \frac{-\tan y + y}{y^2} = -\lim_{\substack{y \to 0 \\ y>0}} \frac{\tan y - y}{y^2} = 0$$

71. Evaluate:

$$\lim_{x\to 0} \frac{\tan x - \arctan x}{x^2}$$

Solution: Using the result from the previous problem, we have:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - \arctan x}{x^2} &= \lim_{x\to 0} \frac{\tan x - x}{x^2} + \lim_{x\to 0} \frac{x - \arctan x}{x^2} \\
&= \lim_{x\to 0} \frac{x - \arctan x}{x^2} \\
&= \lim_{y\to 0} \frac{\tan y - y}{\tan^2 y} \\
&= \lim_{y\to 0} \frac{\tan y - y}{y^2} \cdot \lim_{y\to 0} \frac{y^2}{\tan^2 y} \\
&= 0
\end{aligned}$$

72. Let $a > 0$ and a sequence of real numbers $(x_n)_{n\geq 0}$ such that $x_n \in (0, a)$ and $x_{n+1}(a - x_n) > \dfrac{a^2}{4}$, $(\forall)n \in \mathbb{N}$. Prove that $(x_n)_{n\geq 1}$ is convergent and evaluate $\lim\limits_{n\to\infty} x_n$.

Solution: Rewrite the condition as $\dfrac{x_{n+1}}{a}\left(1 - \dfrac{x_n}{a}\right) > \dfrac{1}{4}$. With the substitution $y_n = \dfrac{x_n}{a}$, we have $y_{n+1}(1 - y_n) > \dfrac{1}{4}$, with $y_n \in (0, 1)$. Then:

$$4y_{n+1} - 4y_n y_{n+1} - 1 > 0 \Leftrightarrow 4y_n y_{n+1} - 4y_{n+1}^2 + 4y_{n+1}^2 - 4y_{n+1} + 1 < 0 \Leftrightarrow 4y_n(y_{n+1} - y_n) > (2y_{n+1} - 1)^2$$

So $y_{n+1} - y_n > 0$, whence the sequence is strictly increasing. Let $\lim\limits_{n\to\infty} y_n = l$. Then $l(1 - l) \geq \dfrac{1}{4} \Leftrightarrow \left(l - \dfrac{1}{2}\right)^2 \leq 0$. Hence $l = \dfrac{1}{2} \Rightarrow \lim\limits_{n\to\infty} x_n = \dfrac{a}{2}$.

73. Evaluate:

$$\lim_{n\to\infty} \cos\left(n\pi\sqrt[2n]{e}\right)$$

Solution: Using $\lim\limits_{f(x)\to 0} \dfrac{a^{f(x)} - 1}{f(x)} = \ln a$, with $a \in \mathbb{R}$, we have:

$$\begin{aligned}
\left|\lim_{n\to\infty} \cos\left(n\pi\sqrt[2n]{e}\right)\right| &= \lim_{n\to\infty} \left|(-1)^n \cdot \cos\left(n\pi\sqrt[2n]{e} - n\pi\right)\right| \\
&= \lim_{n\to\infty} \left|\cos\left(\frac{\pi}{2} \cdot \frac{e^{\frac{1}{2n}} - 1}{\frac{1}{2n}}\right)\right| \\
&= \left|\cos\left(\frac{\pi}{2} \cdot \lim_{n\to\infty} \frac{e^{\frac{1}{2n}} - 1}{\frac{1}{2n}}\right)\right| \\
&= \left|\cos\left(\frac{\pi}{2}\right)\right| \\
&= 0
\end{aligned}$$

It follows that $\lim\limits_{n\to\infty} \cos\left(n\pi\sqrt[2n]{e}\right) = 0$.

74. Evaluate:

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\tan\frac{(n-1)\pi}{2n}}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\tan\frac{(n-1)\pi}{2n}} &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{\frac{1}{n}\tan\frac{(n-1)\pi}{2n}} \\
&= e^{\lim\limits_{n\to\infty}\frac{\tan\frac{(n-1)\pi}{2n}}{n}} \\
&= e^{\lim\limits_{n\to\infty}\frac{\tan\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}{n}} \\
&= e^{\lim\limits_{n\to\infty}\frac{\cot\frac{\pi}{2n}}{n}} \\
&= e^{\lim\limits_{n\to\infty}\frac{1}{n\tan\frac{\pi}{2n}}} \\
&= e^{\lim\limits_{n\to\infty}\frac{2}{\pi}\cdot\frac{\frac{\pi}{2n}}{\tan\frac{\pi}{2n}}} \\
&= e^{\frac{2}{\pi}}
\end{aligned}$$

75. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\binom{n}{k}}$$

Solution: Using AM-GM, we have:

$$\sqrt[n]{n!} = \sqrt[n]{1\cdot 2\cdot 3\cdot\ldots\cdot n} < \frac{1+2+\ldots+n}{n} = \frac{n+1}{2}$$

Therefore $\dfrac{(n+1)^n}{n!} > 2^n \Rightarrow \lim\limits_{n\to\infty}\dfrac{(n+1)^n}{n!} = \infty$. So:

$$\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\binom{n}{k}} = \lim_{n\to\infty}\frac{\prod\limits_{k=1}^{n+1}\binom{n+1}{k}}{\prod\limits_{k=1}^{n}\binom{n}{k}} = \lim_{n\to\infty}\frac{(n+1)^n}{n!} = \infty$$

76. If $a > 0$, evaluate:

$$\lim_{n\to\infty}\frac{a+\sqrt{a}+\sqrt[3]{a}+\ldots+\sqrt[n]{a}-n}{\ln n}$$

Solution:

$$\lim_{n\to\infty} \frac{a+\sqrt{a}+\sqrt[3]{a}+\ldots+\sqrt[n]{a}-n}{\ln n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{a}-1}{\ln(n+1)-\ln n}$$
$$= \lim_{n\to\infty} \frac{n\cdot(\sqrt[n+1]{a}-1)}{\ln\left(1+\frac{1}{n}\right)^n}$$
$$= \lim_{n\to\infty} \left[\frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\frac{n}{n+1}\right]$$
$$= \ln a$$

77. Evaluate:

$$\lim_{n\to\infty} n \ln\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)$$

Solution:

$$\lim_{n\to\infty} n \ln\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right) = \lim_{n\to\infty} \ln\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)^n$$
$$= \ln \lim_{n\to\infty}\left[\left(1+\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1\right)^{\frac{1}{\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1}}\right]^{n\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1\right)}$$
$$= \ln e^{\lim\limits_{n\to\infty}\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1\right)}$$
$$= \lim_{n\to\infty} n\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1\right)$$
$$= \lim_{n\to\infty} n\left(\frac{1+\tan\frac{\pi}{n}}{1-\tan\frac{\pi}{n}}-1\right)$$
$$= \lim_{n\to\infty} \frac{2n\tan\frac{\pi}{n}}{1-\tan\frac{\pi}{n}}$$
$$= 2\lim_{n\to\infty} n\tan\frac{\pi}{n}$$
$$= 2\pi\lim_{n\to\infty} \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$$
$$= 2\pi$$

78. Let $k \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_k \in \mathbb{R}$ such that $a_0+a_1+a_2+\ldots+a_k=0$. Evaluate:

$$\lim_{n\to\infty}\left(a_0\sqrt[3]{n}+a_1\sqrt[3]{n+1}+\ldots+a_k\sqrt[3]{n+k}\right)$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\left(a_0\sqrt[3]{n}+a_1\sqrt[3]{n+1}+\ldots+a_k\sqrt[3]{n+k}\right) &= \lim_{n\to\infty}\left(a_0\sqrt[3]{n}+\sum_{i=1}^{k}a_i\sqrt[3]{n+i}\right)\\
&= \lim_{n\to\infty}\left(-\sqrt[3]{n}\cdot\sum_{i=1}^{k}a_i+\sum_{i=1}^{k}a_i\sqrt[3]{n+i}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{k}a_i\left(\sqrt[3]{n+i}-\sqrt[3]{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{k}\frac{ia_i}{\sqrt[3]{(n+i)^2}+\sqrt[3]{n(n+i)}+\sqrt[3]{n^2}}\\
&= 0
\end{aligned}$$

79. Evaluate:

$$\lim_{n\to\infty}\sin\left(n\pi\sqrt[3]{n^3+3n^2+4n-5}\right)$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\sin\left(n\pi\sqrt[3]{n^3+3n^2+4n-5}\right) &= \lim_{n\to\infty}\sin\left(n\pi\sqrt[3]{n^3+3n^2+4n-5}-n(n+1)\pi\right)\\
&= \lim_{n\to\infty}\sin\left(n\pi\left(\sqrt[3]{n^3+3n^2+4n-5}-n-1\right)\right)\\
&= \lim_{n\to\infty}\sin\left(\frac{n(n-6)\pi}{\sqrt[3]{(n^3-5)^2}+(n+1)\sqrt[3]{n^3-5}+(n+1)^2}\right)\\
&= \sin\left(\pi\lim_{n\to\infty}\frac{1-\frac{6}{n}}{\sqrt[3]{\left(1-\frac{5}{n^3}\right)^2}+\left(1+\frac{1}{n}\right)\sqrt[3]{1-\frac{5}{n^3}}+\left(1+\frac{1}{n}\right)^2}\right)\\
&= \sin\frac{\pi}{3}\\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

80. Evaluate:

$$\lim_{\substack{x\to 1\\ x<1}}\frac{2\arcsin x-\pi}{\sin\pi x}$$

Solution:

$$\begin{aligned}
\lim_{\substack{x \to 1 \\ x<1}} \frac{2\arcsin x - \pi}{\sin \pi x} &= 2 \lim_{\substack{x \to 1 \\ x<1}} \frac{\arcsin x - \frac{\pi}{2}}{\sin\left(\arcsin x - \frac{\pi}{2}\right)} \cdot \lim_{\substack{x \to 1 \\ x<1}} \frac{\sin\left(\arcsin x - \frac{\pi}{2}\right)}{\sin \pi x} \\
&= 2 \lim_{\substack{x \to 1 \\ x<1}} \frac{-\sqrt{1-y^2}}{\sin \pi x} \\
&= -2 \lim_{\substack{y \to 0 \\ y>0}} \frac{\sqrt{y(2-y)}}{\sin \pi(1-y)} \\
&= -2 \lim_{\substack{y \to 0 \\ y>0}} \frac{\sqrt{2(1-y)}}{\sin \pi y} \\
&= -2 \lim_{\substack{y \to 0 \\ y>0}} \frac{\pi y}{\sin \pi y} \cdot \lim_{\substack{y \to 0 \\ y>0}} \frac{\sqrt{2-y}}{\pi\sqrt{y}} \\
&= -\infty
\end{aligned}$$

81. Evaluate:

$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k \ln k}$$

Solution: Using Lagrange formula we can deduce that

$$\frac{1}{k \ln k} > \ln(\ln(k+1)) - \ln(\ln k)$$

Summing from $k = 2$ to n it follows that

$$\sum_{k=2}^{n} \frac{1}{k \ln k} > \ln(\ln(n+1)) - \ln(\ln 2))$$

Then it is obvious that:

$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k \ln k} = \infty$$

82. Evaluate:

$$\lim_{n\to\infty} \left[\lim_{x\to 0} \left(1 + \sum_{k=1}^{n} \sin^2(kx) \right)^{\frac{1}{n^3 x^2}} \right]$$

Solution:

$$\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sum_{k=1}^{n}\sin^2(kx)\right)^{\frac{1}{n^3x^2}}\right] = \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sum_{k=1}^{n}\sin^2(kx)\right)^{\frac{1}{\sum_{k=1}^{n}\sin^2(kx)}}\right]^{\frac{\sum_{k=1}^{n}\sin^2(kx)}{n^3x^2}}$$

$$= \lim_{n\to\infty}\left[e^{\frac{1}{n^3}\lim\limits_{x\to 0}\frac{\sum_{k=1}^{n}\sin^2(kx)}{x^2}}\right]$$

$$= e^{\lim\limits_{n\to\infty}\frac{1^2+2^2+\ldots+n^2}{n^3}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}}$$

$$= \sqrt[3]{e}$$

83. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(k+1)(k+2)\cdot\ldots\cdot(k+p)}{n^{p+1}}$$

Solution: Using Cesaro-Stolz, we have:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(k+1)(k+2)\cdot\ldots\cdot(k+p)}{n^{p+1}} = \sum_{k=0}^{n}\frac{\frac{(k+p)!}{k!}}{n^{p+1}}$$

$$= \lim_{n\to\infty}\frac{\frac{(n+p+1)!}{(n+1)!}}{(n+1)^{p+1}-n^{p+1}}$$

$$= \lim_{n\to\infty}\frac{(n+2)(n+3)\cdot\ldots\cdot(n+p+1)}{n^{p+1}+\binom{p+1}{1}n^p+\ldots+1-n^{p+1}}$$

$$= \lim_{n\to\infty}\frac{n^p+\ldots}{(p+1)n^p+\ldots}$$

$$= \frac{1}{p+1}$$

84. If $\alpha_n \in \left(0, \dfrac{\pi}{4}\right)$ is a root of the equation $\tan\alpha + \cot\alpha = n,\ n \geq 2$, evaluate:

$$\lim_{n\to\infty} (\sin\alpha_n + \cos\alpha_n)^n$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} (\sin\alpha_n + \cos\alpha_n)^n &= \lim_{n\to\infty} \left[(\sin\alpha_n + \cos\alpha_n)^2\right]^{\frac{n}{2}} \\
&= \lim_{n\to\infty} (1 + 2\cos\alpha_n \cdot \sin\alpha_n)^{\frac{n}{2}} \\
&= \lim_{n\to\infty} \left(1 + \frac{2}{\dfrac{\sin^2\alpha_n + \cos^2\alpha_n}{\cos\alpha_n \cdot \sin\alpha_n}}\right)^{\dfrac{n}{2}} \\
&= \lim_{n\to\infty} \left(1 + \frac{2}{\tan\alpha_n + \cot\alpha_n}\right)^{\dfrac{n}{2}} \\
&= \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^{\frac{n}{2}} \\
&= e
\end{aligned}$$

85. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{\dbinom{n+k}{2}}}{n^2}$$

First solution: Cesaro-Stolz gives:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{\dbinom{n+k}{2}}}{n^2} &= \lim_{n\to\infty} \frac{\sqrt{\dbinom{2n+1}{2}} + \sqrt{\dbinom{2n+2}{2}} - \sqrt{\dbinom{n+1}{2}}}{2n+1} \\
&= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{\sqrt{2n(2n+1)} + \sqrt{(2n+1)(2n+2)} - \sqrt{n(n+1)}}{2n+1} \\
&= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{\sqrt{4 + \dfrac{2}{n}} + \sqrt{4 + \dfrac{6}{n} + \dfrac{2}{n^2}} - \sqrt{1 + \dfrac{1}{n}}}{2 + \dfrac{1}{n}} \\
&= \frac{3}{2\sqrt{2}}
\end{aligned}$$

Second solution: Observe that:

$$\binom{n+k}{2} = \frac{(n+k-1)(n+k)}{2} = \frac{n^2}{2}\left(1+\frac{k}{n}\right)\left(1+\frac{k-1}{n}\right)$$

for which we have

$$\frac{n^2}{2}\left(1+\frac{k-1}{n}\right)^2 \leq \frac{n^2}{2}\left(1+\frac{k}{n}\right)\left(1+\frac{k-1}{n}\right) \leq \frac{n^2}{2}\left(1+\frac{k}{n}\right)^2$$

therefore

$$\frac{n}{\sqrt{2}}\left(1+\frac{k-1}{n}\right) \leq \sqrt{\binom{n+k}{2}} \leq \frac{n}{\sqrt{2}}\left(1+\frac{k}{n}\right)$$

Summing from $k=1$ to n, we get:

$$\frac{1}{n\sqrt{2}}\sum_{k=1}\left(1+\frac{k-1}{n}\right) \leq \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{\binom{n+k}{2}}}{n^2} \leq \frac{1}{n\sqrt{2}}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)$$

We can apply the Squeeze theorem because

$$\lim_{n\to\infty}\frac{1}{n\sqrt{2}}\sum_{k=1}\left(1+\frac{k-1}{n}\right) = \lim_{n\to\infty}\frac{1}{n\sqrt{2}}\left(n+\frac{n-1}{2}\right) = \lim_{n\to\infty}\frac{3n-1}{2n\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

and

$$\lim_{n\to\infty}\frac{1}{n\sqrt{2}}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n\sqrt{2}}\left(n+\frac{n+1}{2}\right) = \lim_{n\to\infty}\frac{3n+1}{2n\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

Thus

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{\binom{n+k}{2}}}{n^2} = \frac{3}{2\sqrt{2}}$$

86. Evaluate:

$$\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)}$$

Solution: Using Cesaro-Stolz we'll evaluate:

$$\lim_{n\to\infty} \ln \sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)} = \lim_{n\to\infty} \frac{\displaystyle\sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right)}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n+1} \ln\left(1+\frac{k}{n+1}\right) - \sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln \frac{1+\frac{k}{n+1}}{1+\frac{k}{n}} + \ln 2$$

$$= \lim_{n\to\infty} \ln\left(\frac{4n+2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\right)$$

$$= \ln 4 - 1$$

It follows that:

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)} = 4e^{-1}$$

87. Evaluate:

$$\lim_{x\to 0} \frac{\arctan x - \arcsin x}{x^3}$$

Solution:

$$\lim_{x\to 0} \frac{\arctan x - \arcsin x}{x^3} = \lim_{x\to 0} \frac{\arctan x - \arcsin x}{\tan(\arctan x - \arcsin x)} \cdot \lim_{x\to 0} \frac{\tan(\arctan x - \arcsin x)}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan(\arctan x - \arcsin x)}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{x^3} \cdot \frac{x - \dfrac{x}{\sqrt{1-x^2}}}{1 + \dfrac{x}{\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0} \frac{1}{x^2} \cdot \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+x^2}$$

$$= \lim_{x\to 0} \frac{-1}{(\sqrt{1-x^2}+x^2)(\sqrt{1-x^2}+1)}$$

$$= -\frac{1}{2}$$

88. If $\alpha > 0$, evaluate:

$$\lim_{n\to\infty} \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}}$$

Solution: Let

$$x_n = \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}} = \frac{n^\alpha\left[\left(1+\frac{1}{n}\right)^\alpha - 1\right]}{n^{\alpha-1}} = n\left[\left(1+\frac{1}{n}\right)^\alpha - 1\right]$$

Then $\lim\limits_{n\to\infty} \dfrac{x_n}{n} = 0$. Observe that:

$$1 + \frac{x_n}{n} = \left(1+\frac{1}{n}\right)^\alpha \Leftrightarrow \left[\left(1+\frac{x_n}{n}\right)^{\frac{n}{x_n}}\right]^{x_n} = \left[\left(1+\frac{1}{n}\right)^n\right]^\alpha$$

By passing to limit, we have $e^{\lim\limits_{n\to\infty} x_n} = e^\alpha$. Hence $\lim\limits_{n\to\infty} x_n = \alpha$.

89. Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{2^k}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{2^k} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k(k+1)}{2^k} - \frac{k}{2^k}\right) \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{k^2}{2^{k-1}} - \frac{(k+1)^2}{2^k} + \frac{3k+1}{2^k}\right) - \sum_{k=1}^{n} \frac{k}{2^k}\right] \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{(n+1)^2}{2^n}\right) + \sum_{k=1}^{n} \frac{2k+1}{2^k}\right] \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{(n+1)^2}{2^n}\right) + 2\sum_{k=1}^{n} \frac{k}{2^k} + \sum_{k=1}^{n} \frac{1}{2^k}\right] \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{(n+1)^2}{2^n}\right) + 2\sum_{k=1}^{n} \left(\frac{k}{2^{k-1}} - \frac{k+1}{2^k} + \frac{1}{2^k}\right) + \sum_{k=1}^{n} \frac{1}{2^k}\right] \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{(n+1)^2}{2^n}\right) + 2\left(1 - \frac{n+1}{2^n}\right) + 3\sum_{k=1}^{n} \frac{1}{2^k}\right] \\
&= \lim_{n\to\infty} \left[3 - \frac{n^2+4n+3}{2^n} + 3\left(1 - \frac{1}{2^n}\right)\right] \\
&= \lim_{n\to\infty} \left(6 - \frac{n^2+4n+6}{2^n}\right) \\
&= 6 - \lim_{n\to\infty} \frac{n^2+4n+6}{2^n}
\end{aligned}$$

Because:

$$\lim_{n\to\infty} \frac{\dfrac{(n+1)^2+4(n+1)+6}{2^{n+1}}}{\dfrac{n^2+4n+6}{2^n}} = \lim_{n\to\infty} \frac{n^2+6n+11}{2n^2+8n+12} = \frac{1}{2}$$

it follows that $\lim\limits_{n\to\infty} \dfrac{n^2+4n+6}{2^n} = 0$, therefore our limit is 6.

90. Evaluate:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(k+1)(k+2)}{2^k}$$

Solution: Using the previous limit, we have:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(k+1)(k+2)}{2^k} &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{k^2}{2^k} + 3 \cdot \sum_{k=0}^{n} \frac{k}{2^k} + \sum_{k=0}^{n} \frac{1}{2^{k-1}} \right) \\
&= 6 + 3 \lim_{n\to\infty} \left(2 - \frac{n+2}{2^n} \right) + \lim_{n\to\infty} \left(2 + \frac{1-\frac{1}{2^n}}{\frac{1}{2}} \right) \\
&= 16
\end{aligned}$$

91. Consider a sequence of real numbers $(x_n)_{n\geq 1}$ such that $x_1 \in (0,1)$ and $x_{n+1} = x_n^2 - x_n + 1$, $(\forall) n \in \mathbb{N}$. Evaluate:

$$\lim_{n\to\infty} (x_1 x_2 \cdot \ldots \cdot x_n)$$

Solution: Substracting x_n from both sides of the recurrence formula gives $x_{n+1} - x_n = x_n^2 - 2x_n + 1 = (x_n - 1)^2 \geq 0$ so $(x_n)_{n\geq 1}$ is an increasing sequence.

$x_1 \in (0,1)$ is given as hypothesis. Now if there exists $k \in \mathbb{N}$ such that $x_k \in (0,1)$, then $(x_k - 1) \in (-1, 0)$, so $x_k(x_k - 1) \in (-1, 0)$. Then $x_{k+1} = 1 + x_k(x_k - 1) \in (0,1)$ as well, so by induction we see that the sequence in contained in $(0,1)$.

$(x_n)_{n\geq 1}$ is increasing and bounded from above, so it converges. If $\lim\limits_{n\to\infty} x_n = 1$ then from the recurrence, $l = l^2 - l + 1$ which gives $l = 1$. Thus, $\lim\limits_{n\to\infty} x_n = 1$.

Now rewrite the recurrence formula as $1 - x_{n+1} = x_n(1 - x_n)$. For $n = 1, 2, \ldots, n$, we have:

$$1 - x_2 = x_1(1 - x_1)$$

$$1 - x_3 = x_2(1 - x_2)$$

$$\ldots$$

$$1 - x_n = x_{n-1}(1 - x_{n-1})$$

$$1 - x_{n+1} = x_n(1 - x_n)$$

Multiplying them we have:

$$1 - x_{n+1} = x_1 x_2 \cdot \ldots \cdot x_n(1 - x_1)$$

Thus:

$$\lim_{n\to\infty}(x_1 x_2 \cdot \ldots \cdot x_n) = \lim_{n\to\infty}\frac{1 - x_{n+1}}{1 - x_1} = 0$$

92. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x\to 0}\frac{1 - \cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx}{x^2}$$

Solution: Let

$$a_n = \lim_{x\to 0}\frac{1 - \cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx}{x^2}$$

Then

$$\begin{aligned}
a_{n+1} &= \lim_{x\to 0}\frac{1 - \cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx \cdot \cos(n+1)x}{x^2} \\
&= \lim_{x\to 0}\frac{1 - \cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx}{x^2} + \lim_{x\to 0}\frac{\cos x \cdot \cos 2x \cdot \ldots \cdot nx(1 - \cos(n+1)x)}{x^2} \\
&= a_n + \lim_{x\to 0}\frac{1 - \cos(n+1)x}{x^2} \\
&= a_n + \lim_{x\to 0}\frac{2\sin^2\frac{(n+1)x}{2}}{x^2} \\
&= a_n + \frac{(n+1)^2}{2}\lim_{x\to 0}\left(\frac{\sin\frac{(n+1)x}{2}}{\frac{n+1}{2}}\right)^2 \\
&= a_n + \frac{(n+1)^2}{2}
\end{aligned}$$

Now let $n = 1, 2, 3, \ldots, n-1$:

$$a_0 = 0$$

$$a_1 = a_0 + \frac{1}{2}$$

$$a_2 = a_1 + \frac{2^2}{2}$$

$$a_3 = a_2 + \frac{3^2}{2}$$

$$\dots$$

$$a_n = a_{n-1} + \frac{n^2}{2}$$

Summing gives:

$$a_n = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{n^2}{2} = \frac{1}{2}(1^2 + 2^2 + \ldots + n^2) = \frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6}$$

Finally, the answer is

$$\lim_{x \to 0} \frac{1 - \cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx}{x^2} = \frac{n(n+1)(2n+1)}{12}$$

93. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that x_n is the real root of the equation $x^3 + nx - n = 0,\ n \in \mathbb{N}^*$. Prove that this sequence is convergent and find it's limit.

Solution: Let $f(x) = x^3 + nx - n$. Then $f'(x) = 3x^2 + n > 0$, so f has only one real root which is contained in the interval $(0,1)$(because $f(0) = -n$ and $f(1) = 1$, so $x_n \in (0,1)$).

The sequence $(x_n)_{n \geq 1}$ is strictly increasing, because

$$x_{n+1} - x_n = \frac{1 - x_n}{x_{n+1}^2 + x_{n+1}x_n + x_n^2 + n} > 0$$

Therefore the sequence is convergent. From the equation, we have $x_n = 1 - \frac{x_n^3}{n}$. By passing to limit, we find that $\lim\limits_{n \to \infty} x_n = 1$.

94. Evaluate:

$$\lim_{x \to 2} \frac{\arctan x - \arctan 2}{\tan x - \tan 2}$$

Solution: Using $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \cdot \tan b}$, we have:

$$\begin{aligned}\lim_{x\to 2}\frac{\arctan x-\arctan 2}{\tan x-\tan 2} &= \lim_{x\to 2}\frac{\arctan x-\arctan 2}{\tan(\arctan x-\arctan 2)}\cdot\lim_{x\to 2}\frac{\tan(\arctan x-\arctan 2)}{\tan x-\tan 2}\\ &= \lim_{x\to 2}\frac{\frac{x-2}{1+2x}}{\frac{\sin(x-2)}{\cos x\cdot\cos 2}}\\ &= \lim_{x\to 2}\frac{x-2}{\sin(x-2)}\cdot\lim_{x\to 2}\frac{\cos x\cdot\cos 2}{1+2x}\\ &= \lim_{x\to 2}\frac{\cos x\cdot\cos 2}{1+2x}\\ &= \frac{\cos^2 2}{5}\end{aligned}$$

95. Evaluate:

$$\lim_{n\to\infty}\frac{1+\sqrt[2^2]{2!}+\sqrt[3^2]{3!}+\ldots+\sqrt[n^2]{n!}}{n}$$

Solution: Using Cesaro-Stolz:

$$\lim_{n\to\infty}\frac{1+\sqrt[2^2]{2!}+\sqrt[3^2]{3!}+\ldots+\sqrt[n^2]{n!}}{n}=\lim_{n\to\infty}\sqrt[(n+1)^2]{(n+1)!}$$

Also, an application of AM-GM gives:

$$\begin{aligned}1 &\le \sqrt[(n+1)^2]{(n+1)!}\\ &= \sqrt[n+1]{\sqrt[n+1]{1\cdot 2\cdot 3\cdot\ldots\cdot n\cdot(n+1)}}\\ &< \sqrt[n+1]{\frac{1+2+3+\ldots+n+n+1}{n+1}}\\ &= \sqrt[n+1]{\frac{n+2}{2}}\end{aligned}$$

Thus

$$1\le\lim_{n\to\infty}\sqrt[(n+1)^2]{(n+1)!}\le\lim_{n\to\infty}\sqrt[n+1]{\frac{n+2}{2}}=1$$

From the Squeeze Theorem it follows that:

$$\lim_{n\to\infty}\frac{1+\sqrt[2^2]{2!}+\sqrt[3^2]{3!}+\ldots+\sqrt[n^2]{n!}}{n}=1$$

96. Let $(x_n)_{n\ge 1}$ such that $x_1>0$, $x_1+x_1^2<1$ and $x_{n+1}=x_n+\dfrac{x_n^2}{n^2}$, $(\forall)n\ge 1$. Prove that the sequences $(x_n)_{n\ge 1}$ and $(y_n)_{n\ge 2}$, $y_n=\dfrac{1}{x_n}-\dfrac{1}{n-1}$ are convergent.

Solution: $x_{n+1} - x_n = \dfrac{x_n^2}{n^2}$, so the $(x_n)_{n\geq 1}$ is strictly increasing.

$$x_2 = x_1 + x_1^2 < 1 \Rightarrow \frac{1}{x_2} > 1 \Rightarrow y_2 = \frac{1}{x_2} - 1 > 0$$

Also

$$\begin{aligned} y_{n+1} - y_n &= \frac{1}{x_{n+1}} - \frac{1}{n} - \frac{1}{x_n} + \frac{1}{n-1} \\ &= \frac{1}{n(n-1)} - \frac{x_{n+1} - x_n}{x_n x_{n+1}} \\ &= \frac{1}{n(n-1)} - \frac{x_n}{n^2 x_{n+1}} \\ &> \frac{1}{n(n-1)} - \frac{1}{n^2} \\ &= \frac{1}{n^2(n-1)} \\ &> 0 \end{aligned}$$

Hence $(y_n)_{n\geq 2}$ is strictly increasing. Observe that $x_n = \dfrac{1}{y_n + \dfrac{1}{n-1}}$. So $\lim\limits_{n\to\infty} x_n = \dfrac{1}{\lim\limits_{n\to\infty} y_n}$. Assuming that $\lim\limits_{n\to\infty} y_n = \infty$, we have $\lim\limits_{n\to\infty} x_n = 0$, which is a contradiction, because $x_1 > 0$ and the sequence $(x_n)_{n\geq 1}$ is strictly increasing. Hence $(y_n)_{n\geq 2}$ is convergent. It follows that $(x_n)_{n\geq 2}$ is also convergent.

97. Evaluate:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin \frac{2i}{n^2}$$

First solution: Let's start from

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Leftrightarrow (\forall)\varepsilon > 0,\ (\exists)\delta > 0,\ (\forall)x \in (-\delta, \delta)\backslash\{0\} \Rightarrow \left|\frac{\sin x}{x} - 1\right| < \varepsilon$$

Let some arbitrary $\varepsilon > 0$. For such ε, $(\exists)\delta > 0$ such that $(\forall)x \in (-\delta, \delta)\backslash\{0\}$, we have $1 - \varepsilon < \dfrac{\sin x}{x} < 1 + \varepsilon$. For $\delta > 0$, $(\exists)n_\varepsilon \in \mathbb{N}^*$ such that $\dfrac{2}{n} < \delta$, $(\forall)n \geq n_\varepsilon$. Because $0 < \dfrac{2i}{n^2} \leq \dfrac{2}{n}$, $(\forall)1 \leq i \leq n,\ n \geq n_\varepsilon$, we have:

$$1 - \varepsilon < \frac{\sin \dfrac{2i}{n^2}}{\dfrac{2i}{n^2}} < 1 + \varepsilon$$

Summing, we get:

$$(1-\varepsilon)\sum_{i=1}^{n}\frac{2i}{n^2} < \sum_{i=1}^{n}\sin\frac{2i}{n^2} < (1+\varepsilon)\sum_{i=1}^{n}\frac{2i}{n^2}$$

Or equivalently:

$$\frac{(1-\varepsilon)(n+1)}{n} < \sum_{i=1}\sin\frac{2i}{n^2} < \frac{(1+\varepsilon)(n+1)}{n}$$

By passing to limit:

$$1-\varepsilon \leq \lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{2i}{n^2} \leq 1+\varepsilon$$

Or

$$\left|\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{2i}{n^2} - 1\right| \leq \varepsilon,\ (\forall)\varepsilon > 0$$

which implies that:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{2i}{n^2} = 1$$

Second solution: Start with the formula

$$\sum_{i=1}^{n}\sin(x+yi) = \frac{\sin\dfrac{(n+1)y}{2}\cdot\sin\left(x+\dfrac{ny}{2}\right)}{\sin\dfrac{y}{2}}$$

Setting $x=0,\ y=\dfrac{2}{n^2}$, it rewrites as

$$\sum_{i=1}^{n}\sin\frac{2i}{n^2} = \frac{\sin\dfrac{n+1}{n^2}\sin\dfrac{1}{n}}{\sin\dfrac{1}{n^2}}$$

whence

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{2i}{n^2} = \lim_{n\to\infty}\frac{\dfrac{\sin\frac{n+1}{n^2}}{\frac{n+1}{n^2}}\cdot\dfrac{\sin\frac{1}{n}}{\frac{1}{n}}}{\dfrac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}}\cdot\lim_{n\to\infty}\frac{n+1}{n} = 1$$

98. If $a>0,\ a\neq 1$, evaluate:

$$\lim_{x\to a}\frac{x^x - a^x}{a^x - a^a}$$

Solution: As $\lim\limits_{x\to a} x\ln\dfrac{x}{a} = 0$, we have:

$$\begin{aligned}
\lim_{x\to a}\frac{x^x - a^x}{a^x - a^a} &= \lim_{x\to a}\frac{e^{x\ln x} - e^{x\ln a}}{a^x - a^a}\\
&= \lim_{x\to a}\frac{e^{x\ln a}\left(e^{x\ln\frac{x}{a}} - 1\right)}{a^a(a^{x-a} - 1)}\\
&= \left(\lim_{x\to a}\frac{e^{x\ln a}}{a^a}\right)\cdot\left(\lim_{x\to a}\frac{e^{x\ln\frac{x}{a}} - 1}{x\ln\frac{x}{a}}\right)\cdot\left(\lim_{x\to a}\frac{x-a}{a^{x-a}-1}\right)\cdot\left(\lim_{x\to a}\frac{x\ln\frac{x}{a}}{x-a}\right)\\
&= \frac{1}{\ln a}\cdot\lim_{x\to a}\left[x\ln\left(\frac{x}{a}\right)^{\frac{1}{x-a}}\right]\\
&= \frac{a}{\ln a}\cdot\lim_{x\to a}\left[\left(1+\frac{x-a}{a}\right)^{\dfrac{a}{x-a}}\right]^{\dfrac{1}{a}}\\
&= \frac{a}{\ln a}\cdot\ln e^{\frac{1}{a}}\\
&= \frac{1}{\ln a}
\end{aligned}$$

99. Consider a sequence of positive real numbers $(a_n)_{n\geq 1}$ such that $a_{n+1} - \dfrac{1}{a_{n+1}} = a_n + \dfrac{1}{a_n}$, $(\forall)n \geq 1$. Evaluate:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

Solution: $(a_n)_{n\geq 1}$ is clearly an increasing sequence. If it has a finite limit, say l, then

$$l - \frac{1}{l} = l + \frac{1}{l} \Rightarrow \frac{2}{l} = 0$$

contradiction. Therefore a_n approaches infinity. Let $y_n = \dfrac{1}{a_n^2} + a_n^2$. Then $y_{n+1} = y_n + 4$. So

$$y_2 = y_1 + 4$$

$$y_3 = y_2 + 4$$

$$\ldots$$

$$y_{n+1} = y_n + 4$$

Summing, it results that $y_{n+1} = y_1 + 4n$, which rewrites as

$$a_{n+1}^2 + \frac{1}{a_{n+1}^2} = y_1 + 4n \Leftrightarrow \left(a_{n+1} + \frac{1}{a_{n+1}}\right)^2 = y_1 + 2 + 4n \Leftrightarrow$$

$$a_{n+1} + \frac{1}{a_{n+1}} = \sqrt{4n + y_1 + 2} \Rightarrow a_{n+1}^2 - \sqrt{4n + y_1 + 2} \cdot a_{n+1} + 1 = 0$$

from which $a_{n+1} = \dfrac{\sqrt{4n + y_1 + 2} \pm \sqrt{4n + y_1 - 2}}{2}$. If we accept that $a_{n+1} = \dfrac{\sqrt{4n + y_1 + 2} - \sqrt{4n + y_1 - 2}}{2}$, then:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{\sqrt{4n + y_1 + 2} - \sqrt{4n + y_1 - 2}}{2} = \lim_{n\to\infty} \frac{2}{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}} = 0$$

which is false, therefore $a_{n+1} = \dfrac{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}}{2}$.

By Cesaro-Stolz, we obtain:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) &= \lim_{n\to\infty} \frac{\frac{1}{a_n}}{\sqrt{n+1} - \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n} + \sqrt{n+1}}{a_{n+1}} \\
&= \lim_{n\to\infty} \frac{2(\sqrt{n} + \sqrt{n+1})}{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}} \\
&= \lim_{n\to\infty} \frac{2(1 + \sqrt{1 + \frac{1}{n}})}{\sqrt{4 + \frac{y_1}{n} + \frac{2}{n}} + \sqrt{4 + \frac{y_1}{n} - \frac{2}{n}}} \\
&= 1
\end{aligned}$$

100. Evaluate:

$$\lim_{x\to 0} \frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\frac{2^{\arctan x}-2^{\arcsin x}}{2^{\tan x}-2^{\sin x}} &= \lim_{x\to 0}\frac{2^{\arcsin x}(2^{\arctan x-\arcsin x}-1)}{2^{\sin x}(2^{\tan x-\sin x}-1)}\\
&= \lim_{x\to 0}\frac{2^{\arctan x-\arcsin x}-1}{2^{\tan x-\sin x}-1}\\
&= \lim_{x\to 0}\frac{2^{\arctan x-\arcsin x}-1}{\arctan x-\arcsin x}\cdot\lim_{x\to 0}\frac{\tan x-\sin x}{2^{\tan x-\sin x}-1}\cdot\lim_{x\to 0}\frac{\arctan x-\arcsin x}{\tan x-\sin x}\\
&= \ln 2\cdot\frac{1}{\ln 2}\cdot\lim_{x\to 0}\frac{\arctan x-\arcsin x}{\tan x-\sin x}\\
&= \lim_{x\to 0}\frac{\arctan x-\arcsin x}{x^3}\cdot\lim_{x\to 0}\frac{x^3}{\tan x-\sin x}\\
&= \lim_{x\to 0}\frac{\arctan x-\arcsin x}{\tan(\arctan x-\arcsin x)}\cdot\lim_{x\to 0}\frac{\tan(\arctan x-\arcsin x)}{x^3}\cdot\lim_{x\to 0}\frac{x^3}{\tan x(1-\cos x)}\\
&= \lim_{x\to 0}\frac{\dfrac{x-\frac{x}{\sqrt{1-x^2}}}{1+\frac{x^2}{\sqrt{1-x^2}}}}{x^3}\cdot\lim_{x\to 0}\frac{x^3}{2\tan x\cdot\sin^2\frac{x}{2}}\\
&= \lim_{x\to 0}\frac{\sqrt{1-x^2}-1}{x^2(\sqrt{1-x^2}+x^2)}\cdot\lim_{x\to 0}\frac{x}{\tan x}\cdot 2\lim_{x\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2\\
&= 2\lim_{x\to 0}\frac{-x^2}{x^2(\sqrt{1-x^2}+x^2)(\sqrt{1-x^2}+1)}\\
&= -1
\end{aligned}$$