Trigonometry Problems

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1. Prove that:

$$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}=\frac{\sqrt{13}-1}{4}$$

2. Prove that $2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right)$ is a root of the equation:

$$\sqrt{4+\sqrt{4+\sqrt{4-x}}}=x$$

3. Prove that

$$\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos 8\theta}}}=\cos\theta$$

4. Prove that

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}$$

5. Prove that

$$\cos x\cdot\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{4}\right)\cdot\cos\left(\frac{x}{8}\right)=\frac{\sin 2x}{16\sin\left(\frac{x}{8}\right)}$$

6. Prove that

$$64\cdot\sin 10^\circ\cdot\sin 20^\circ\cdot\sin 30^\circ\cdot\sin 40^\circ\cdot\sin 50^\circ\cdot\sin 60^\circ\cdot\sin 70^\circ\cdot\sin 80^\circ\cdot\sin 90^\circ=\frac{3}{4}$$

7. Find x if

$$\sin x = \tan 12^\circ \cdot \tan 48^\circ \cdot \tan 54^\circ \cdot \tan 72^\circ \cdot$$

8. Solve the following equations in $\mathbb{R}$:

- $\sin 9x + \sin 5x + 2\sin^2 x = 1$
- $\cos 5x \cdot \cos 3x - \sin 3x \cdot \sin x = \cos 2x$
- $\cos 5x + \cos 3x + \sin 5x + \sin 3x = 2 \cdot \cos\left(\frac{\pi}{4} - 4x\right)$
- $\sin x + \cos x - \sin x \cdot \cos x = -1$
- $\sin 2x - \sqrt{3}\cos 2x = 2$

9. Prove following equations:

- $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) - \sin\left(\frac{6\pi}{7}\right) = 4\sin\left(\frac{\pi}{7}\right) \cdot \sin\left(\frac{3\pi}{7}\right) \cdot \sin\left(\frac{5\pi}{7}\right)$
- $\cos\left(\frac{\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right) + \cos\left(\frac{7\pi}{13}\right) + \cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{11\pi}{13}\right) = \frac{1}{2}$
- $\forall k \in \mathbb{N} \ : \ \cos\left(\frac{\pi}{2k+1}\right) + \cos\left(\frac{3\pi}{2k+1}\right) + \cdots + \cos\left(\frac{(2k-1)\pi}{2k+1}\right) = \frac{1}{2}$
- $\sin\left(\frac{\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) = \dfrac{1}{4} \cdot \cot\left(\frac{\pi}{4}\right)$

10. Show that

$$\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n} = -1.$$

11. Show that $\cos a + \cos 3a + \cos 5a + \cdots + \cos(2n-1)a = \frac{\sin 2na}{2\sin a}$.

12. Show that $\sin a + \sin 3a + \sin 5a + \cdots + \sin(2n-1)a = \frac{\sin^2 na}{\sin a}$.

13. Calculate

$$(\tan 1^\circ)^2 + (\tan 2^\circ)^2 + (\tan 3^\circ)^2 + \ldots + (\tan 89^\circ)^2.$$

14. Prove that $\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7} = 5$.

15. Show that $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}$.

16. $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$ and $\cos\left(\frac{6\pi}{7}\right)$ are the roots of an equation of the form $ax^3+bx^2+cx+d=0$ where a,b,c,d are integers. Determine a,b,c and d.

***17.** Find the value of the sum

$$\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{6\pi}{7}}.$$

18. Solve the equation

$$2\sin^4 x(\sin 2x-3)-2\sin^2 x(\sin 2x-3)-1=0.$$

19. Express the sum of the following series in terms of $\sin x$ and $\cos x$.

$$\sum_{k=0}^{n}(2k+1)\sin^2\left(x+\frac{k}{2}\pi\right)$$

20. Find the smallest positive integer N for which

$$\frac{1}{\sin 45^\circ\cdot\sin 46^\circ}+\frac{1}{\sin 47^\circ\cdot\sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ\cdot\sin 134^\circ}=\frac{1}{\sin N^\circ}.$$

21. Find the value of

$$\frac{\sin 40^\circ+\sin 80^\circ}{\sin 110^\circ}.$$

22. Evaluate the sum

$$S=\tan 1^\circ\cdot\tan 2^\circ+\tan 2^\circ\cdot\tan 3^\circ+\tan 3^\circ\cdot\tan 4^\circ+\cdots+\tan 2004^\circ\cdot\tan 2005^\circ.$$

23. Solve the equation :

$$\sqrt{3}\sin x(\cos x-\sin x)+(2-\sqrt{6})\cos x+2\sin x+\sqrt{3}-2\sqrt{2}=0.$$

24. Let $f(x)=\frac{1}{\sin\frac{\pi x}{7}}$. Prove that $f(3)+f(2)=f(1)$.

25. Suppose that real numbers x,y,z satisfy

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}=p$$

Prove that

$$\cos(x+y)+\cos(y+z)+\cos(x+z)=p.$$

26. Solve for $\theta, 0 \leq \theta \leq \frac{\pi}{2}$:

$$\sin^5\theta+\cos^5\theta=1.$$

27. For $x, y \in [0, \frac{\pi}{3}]$ prove that $\cos x + \cos y \leq 1 + \cos xy$.

28. Prove that among any four distinct numbers from the interval $(0, \frac{\pi}{2})$ there are two, say x, y, such that:

$$8\cos x\cos y\cos(x-y)+1>4(\cos^2 x+\cos^2 y).$$

29. Let $B=\frac{\pi}{7}$. Prove that

$$\tan B\cdot\tan 2B+\tan 2B\cdot\tan 4B+\tan 4B\cdot\tan B=-7.$$

30. a) Calculate

$$\frac{1}{\cos\frac{6\pi}{13}}-4\cos\frac{4\pi}{13}-4\cos\frac{5\pi}{13}=?$$

b) Prove that

$$\tan\frac{\pi}{13}+4\sin\frac{4\pi}{13}=\tan\frac{3\pi}{13}+4\sin\frac{3\pi}{13}$$

c) Prove that

$$\tan\frac{2\pi}{13}+4\sin\frac{6\pi}{13}=\tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13}$$

31. Prove that if α, β are angles of a triangle and $\left(\cos^2\alpha+\cos^2\beta\right)(1+\tan\alpha\cdot\tan\beta)=2$, then $\alpha+\beta=90°$.

32. Let $a, b, c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be real numbers such that $\sin a+\sin b+\sin c+\sin d=1$ and $\cos 2a+\cos 2b+\cos 2c+\cos 2d \geq \frac{10}{3}$. Prove that $a, b, c, d \in [0, \frac{\pi}{6}]$

33. Find all integers m, n for which we have $\sin^m x+\cos^n x=1$, for all x.

34. Prove that $\tan 55°\cdot\tan 65°\cdot\tan 75°=\tan 85°$.

35. Prove that $\frac{4\cos 12°+4\cos 36°+1}{\sqrt{3}}=\tan 78°$.

36. Prove that

$$\sqrt{4+\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4+\sqrt{4-\cdots}}}}}} = 2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right).$$

The signs: $++-++-++-++-\cdots$

37. For reals x, y Prove that $\cos x+\cos y+\sin x\sin y \leq 2$.

38. Solve the equation in real numbers

$$\sqrt{7+2\sqrt{7-2\sqrt{7-2x}}} = x.$$

39. Let A, B, C be three angles of triangle ABC. Prove that

$$(1-\cos A)(1-\cos B)(1-\cos C) \geq \cos A\cos B\cos C.$$

40. Solve the equation

$$\sin^3(x)-\cos^3(x) = \sin^2(x).$$

41. Find $S_n = \sum_{k=1}^{n}\sin^2 k\theta$ for $n \geqslant 1$

42. Prove the following without using induction:

$$\cos x+\cos 2x+\cdots+\cos nx = \frac{\cos\frac{n+1}{2}x\cdot\sin\frac{n}{2}x}{\sin\frac{x}{2}}.$$

43. Evaluate:

$$\sin\theta+\frac{1}{2}\cdot\sin 2\theta+\frac{1}{2^2}\cdot\sin 3\theta+\frac{1}{2^3}\cdot\sin 4\theta+\cdots$$

44. Compute

$$\sum_{k=1}^{n-1}\csc^2\left(\frac{k\pi}{n}\right).$$

45. Prove that

- $\tan\theta+\tan\left(\theta+\frac{\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)+\cdots+\tan\left[\theta+\frac{(n-1)\pi}{n}\right] = -n\cot\left(n\theta+\frac{n\pi}{2}\right).$

- $\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + \frac{2\pi}{n}\right) + \cdots + \cot\left[\theta + \frac{(n-1)\pi}{n}\right] = n\cot n\theta.$

46. Calculate

$$\sum_{n=1}^{\infty} 2^{2n}\sin^4\frac{a}{2^n}.$$

47. Compute the following sum:

$$\tan 1^\circ + \tan 5^\circ + \tan 9^\circ + \cdots + \tan 177^\circ.$$

48. Show that for any positive integer $n > 1$,

- $\displaystyle\sum_{k=0}^{n-1}\cos\frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}(1 + \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2})$
- $\displaystyle\sum_{k=0}^{n-1}\sin\frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}(1 + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2})$

49. Evaluate the product

$$\prod_{k=1}^{n} tan\frac{k\pi}{2(n+1)}.$$

50. Prove that, $\displaystyle\sum_{k=1}^{n}(-1)^{k-1}\cot\frac{(2k-1)\pi}{4n} = n$ for even n.

51. Prove that $\displaystyle\sum_{k=1}^{n}\cot^2\left\{\frac{(2k-1)\pi}{2n}\right\} = n(2n-1).$

52. Prove that $\displaystyle\sum_{k=1}^{n}\cot^4\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)(4n^2+10n-9)}{45}.$

53. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum $\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$ is positive.