

12 已知 $f(x)$ 是一個三次實係數多項式，且 x_1, x_2, x_3, x_4, x_5 為等差數列，試證：

$$C_0^4 f(x_1) - C_1^4 f(x_2) + C_2^4 f(x_3) - C_3^4 f(x_4) + C_4^4 f(x_5) = 0$$

$$\begin{aligned} f(x) &= f(x_1) \times \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + f(x_2) \times \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ &+ f(x_3) \times \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + f(x_4) \times \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \end{aligned}$$

$\because x_1, x_2, x_3, x_4, x_5$ 是等差數列，令公差為 d

$$\begin{aligned} f(x_5) &= f(x_1) \times \frac{3d \times 2d \times d}{(-d) \times (-2d) \times (-3d)} + f(x_2) \times \frac{4d \times 2d \times d}{d \times (-d) \times (-2d)} \\ &+ f(x_3) \times \frac{4d \times 3d \times d}{2d \times d \times (-d)} + f(x_4) \times \frac{4d \times 3d \times 2d}{3d \times 2d \times d} \\ &= -f(x_1) + 4f(x_2) - 6f(x_3) + 4f(x_4) \end{aligned}$$

$$C_0^4 f(x_1) - C_1^4 f(x_2) + C_2^4 f(x_3) - C_3^4 f(x_4) + C_4^4 f(x_5) = 0$$