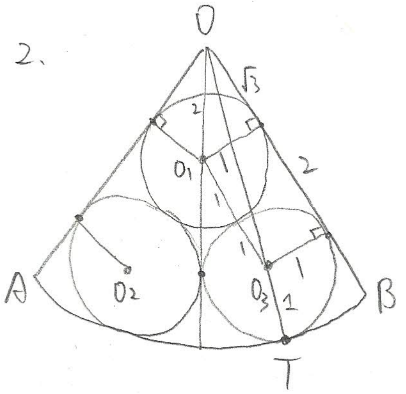
2012 TRML Team　　　　by Fermat 2012.08.21

1. $x^2+3x-10=0 \Rightarrow (x+5)(x-2)=0 \Rightarrow x=-5 \vee 2$

$\Rightarrow$ 交点 $(-5,\frac{25}{4}),\ (2,1)$, $O(0,0)$

$\Rightarrow \triangle OAB = \frac{1}{2}\left|\begin{vmatrix}-5 & \frac{25}{4}\\ 2 & 1\end{vmatrix}\right| = \frac{1}{2}\left|-5-\frac{25}{2}\right| = \frac{1}{2}\cdot\frac{35}{2} = \frac{35}{4}$ ※

2.



$\overline{OO_3} = \sqrt{1^2+(2+\sqrt{3})^2} = \sqrt{8+2\sqrt{12}} = \sqrt{6}+\sqrt{2}$

∴ 扇形半径 $r = \overline{OO_3}+\overline{O_3T}$

$= \sqrt{6}+\sqrt{2}+1$

∴ $a+b+c = 6+2+1 = 9$ ※

3. $2\times 3^2 \mid n \Rightarrow n = 2^a\cdot 3^b$, $a\geq 1$, $b\geq 2$, $a, b\in\mathbb{N}$

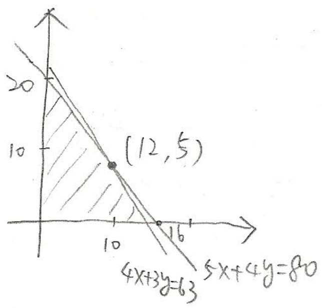
$\Rightarrow (a+1)(b+1) \geq \frac{n}{18} = 2^{a-1}\cdot 3^{b-2}$

a	1	2	3	4	5↑
b	2~4	2~3	2~3	2~3	×

∴ 共 $3+2+2+2=9$ 個 ※

4. 設大套間 x 間，小套間 y 間

$$\begin{cases} 8x+6y\leq 126 \\ x+0.8y\leq 16 \\ x\geq 0,\ y\geq 0\end{cases} \Rightarrow \begin{cases} 4x+3y\leq 63 \\ 5x+4y\leq 80 \\ x\geq 0,\ y\geq 0\end{cases}$$

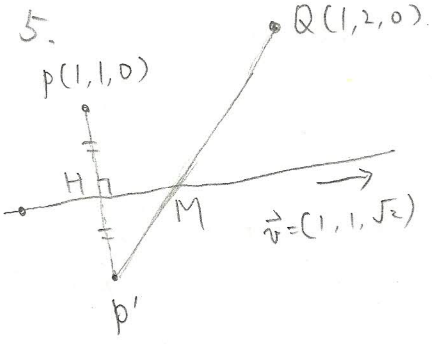


則 $f(x,y) = \underbrace{1.6x+1.2y}_{m=-\frac{4}{3}}$ 之 Max 為 $f(12,5)$

$= 19.2+6$

$= 25.2$ 萬元 $= 252000$ 元 ※

5.



設 $H(1+t,\ t,\ \sqrt{2}t)$

$\Rightarrow \overrightarrow{PH} = (t,\ t-1,\ \sqrt{2}t) \perp (1,1,\sqrt{2})$

$\Rightarrow 2t-1+2t=0$, $t=\frac{1}{4} \Rightarrow H(\frac{5}{4},\frac{1}{4},\frac{\sqrt{2}}{4}),\ P'(\frac{3}{2},-\frac{1}{2},\frac{\sqrt{2}}{2})$

∴ min $= \overline{P'Q} = \sqrt{(\frac{1}{2})^2+(\frac{5}{2})^2+(\frac{\sqrt{2}}{2})^2} = \sqrt{\frac{28}{4}} = \sqrt{7}$ ※

6. $\begin{bmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix}^{100} = 2^{100}\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}^{100} = 2^{100}\begin{bmatrix}\cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3}\\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3}\end{bmatrix}^{100}$

$= 2^{100}\begin{bmatrix}\cos\frac{400}{3}\pi & -\sin\frac{400}{3}\pi\\ \sin\frac{400}{3}\pi & \cos\frac{400}{3}\pi\end{bmatrix} = 2^{100}\begin{bmatrix}\cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3}\\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3}\end{bmatrix} = \begin{bmatrix}-2^{99} & 2^{99}\sqrt{3}\\ -2^{99}\sqrt{3} & -2^{99}\end{bmatrix}$

$\therefore \log_2 \frac{bc-ad}{a+b+c+d} = \log_2 \frac{-3\cdot 2^{198} - 2^{198}}{-2^{100}} = \log_2 2^{100} = 100$ ※

7. 設原有n個車站，增設m個車站 $(m \geq 2)$

$\Rightarrow (n+m)(n+m-1) - n(n-1) = 52$

$\Rightarrow \cancel{n^2} + (2m-1)n + m(m-1) - \cancel{n^2} + n - 52 = 0$

$\Rightarrow m^2 + (2n-1)m - 52 = 0 \quad , m, n \in \mathbb{N}, m \geq 2$

由有理根定理知 $m \mid 52$ $\Rightarrow$

m	2	$\boxed{4}$	13	52
n	$\frac{45}{2}$	$\boxed{5}$	<0	<0

$\therefore n+m = 5+4 = 9$ 個 ※

8. $f\left(\frac{t}{1+t}\right) + f\left(\frac{1+t}{t}\right)\log(1+t) = f\left(\frac{1+t}{t}\right)\log t + 2012 \quad , t>0$

$\Rightarrow f\left(\frac{1+t}{t}\right)\log\left(\frac{1+t}{t}\right) + f\left(\frac{t}{1+t}\right) = 2012$ ，令 $x = \frac{1+t}{t} > 1 > 0$

$\Rightarrow f(x)\log x + f\left(\frac{1}{x}\right) = 2012$ —① ，將 x 以 $\frac{1}{x}$ 代入

$f\left(\frac{1}{x}\right)\log\frac{1}{x} + f(x) = 2012$

相減得 $f(x)(\log x - 1) + f\left(\frac{1}{x}\right)\underbrace{\left(1 - \log\frac{1}{x}\right)}_{= 1+\log x} = 0 \Rightarrow f\left(\frac{1}{x}\right) = \frac{1-\log x}{1+\log x} f(x)$ 代入①

得 $f(x)\cdot\left(\log x + \frac{1-\log x}{1+\log x}\right) = 2012 \Rightarrow f(1000)\cdot\underbrace{\left(3 + \frac{1-3}{1+3}\right)}_{\frac{5}{2}} = 2012$

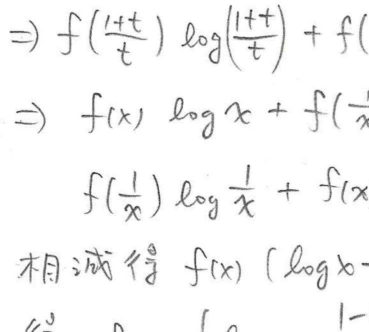
$\Rightarrow f(1000) = 2012 \times \frac{2}{5} = \frac{4024}{5}$ ※

9. 設第i列第j行為 a_{ij} $\Rightarrow a_{ij} = 2i-1 + (j-1)(i+1) = ij+i+j-2 = 2012$

$\Rightarrow \underbrace{(i+1)}_{\geq 2}\underbrace{(j+1)}_{\geq 2} = 2015 = 5\times 13\times 31$ 有 $2\times 2\times 2 = 8$ 個正因數，扣除1及2015

得 $8-2 = 6$ 個 ※

10.



設 $P(5t, 2t)$, $Q(3s, -2s)$, $0<s,t<1$

$\Rightarrow \begin{cases}\triangle AEP = \frac{1}{2}\triangle ACD = \frac{1}{2}\cdot 6 = 3\\ \triangle AEQ = \frac{1}{2}\triangle ABC = \frac{1}{2}\cdot 6 = 3\end{cases} \Rightarrow \overline{PE} = \overline{QE}$

$\Rightarrow E\left(\frac{5t+3s}{2}, \frac{2t-2s}{2}\right) \underset{s=t}{=} (4t, 0)$

$\Rightarrow \triangle AEP = \frac{1}{2}\cdot 4t\cdot 2t = 3 \Rightarrow t = \frac{\sqrt{3}}{2} \quad \therefore 4t = 2\sqrt{3}$ ※